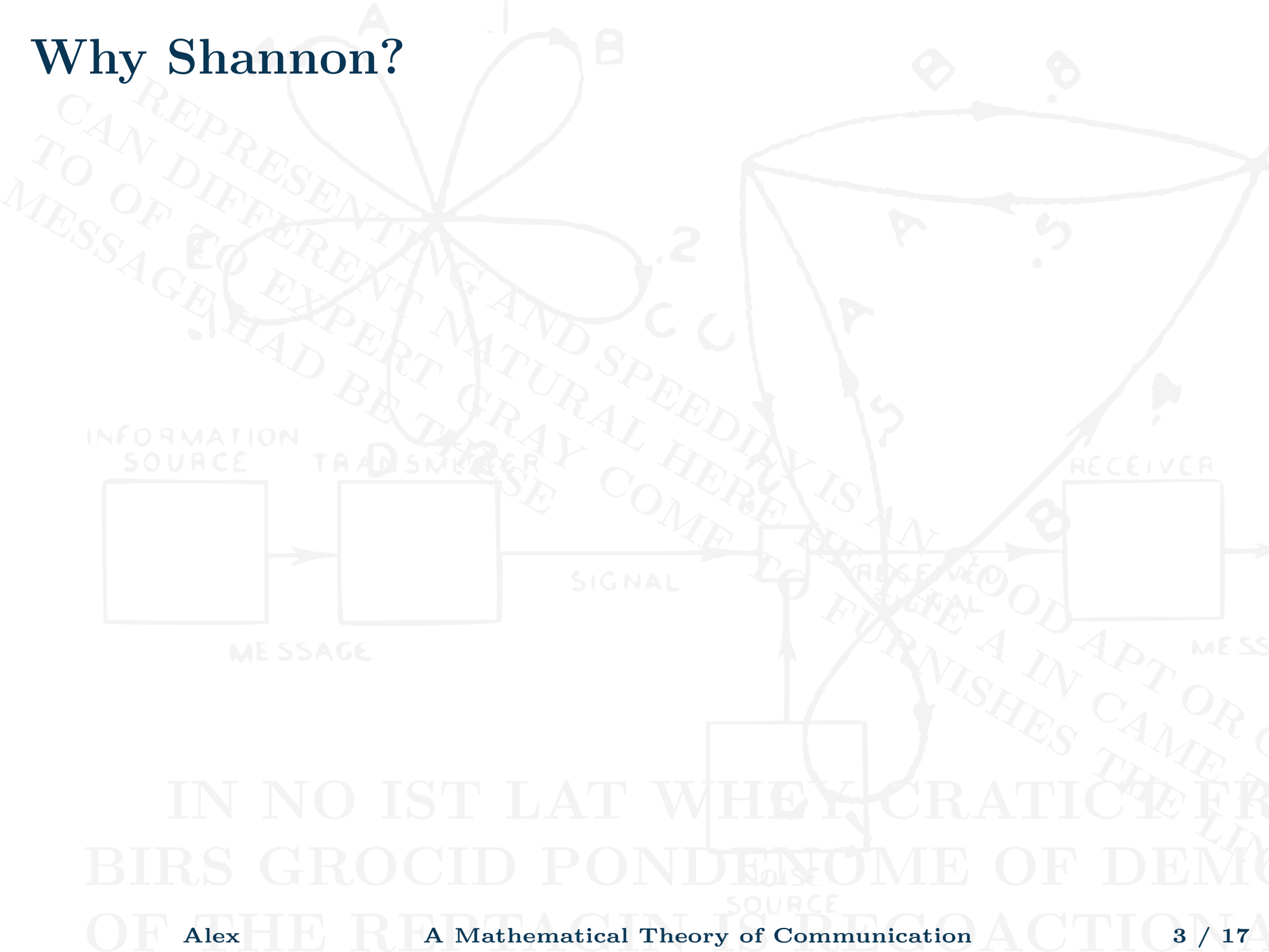
A Mathematical Theory of Communication

(after C. E. Shannon)

Alex Vlasiuk

Image: IEEE Information Theory Society

Why Shannon?


INFORMATION SOURCE
TRANSMITTER
SIGNAL
RECEIVER
MESSAGE
NOISE SOURCE

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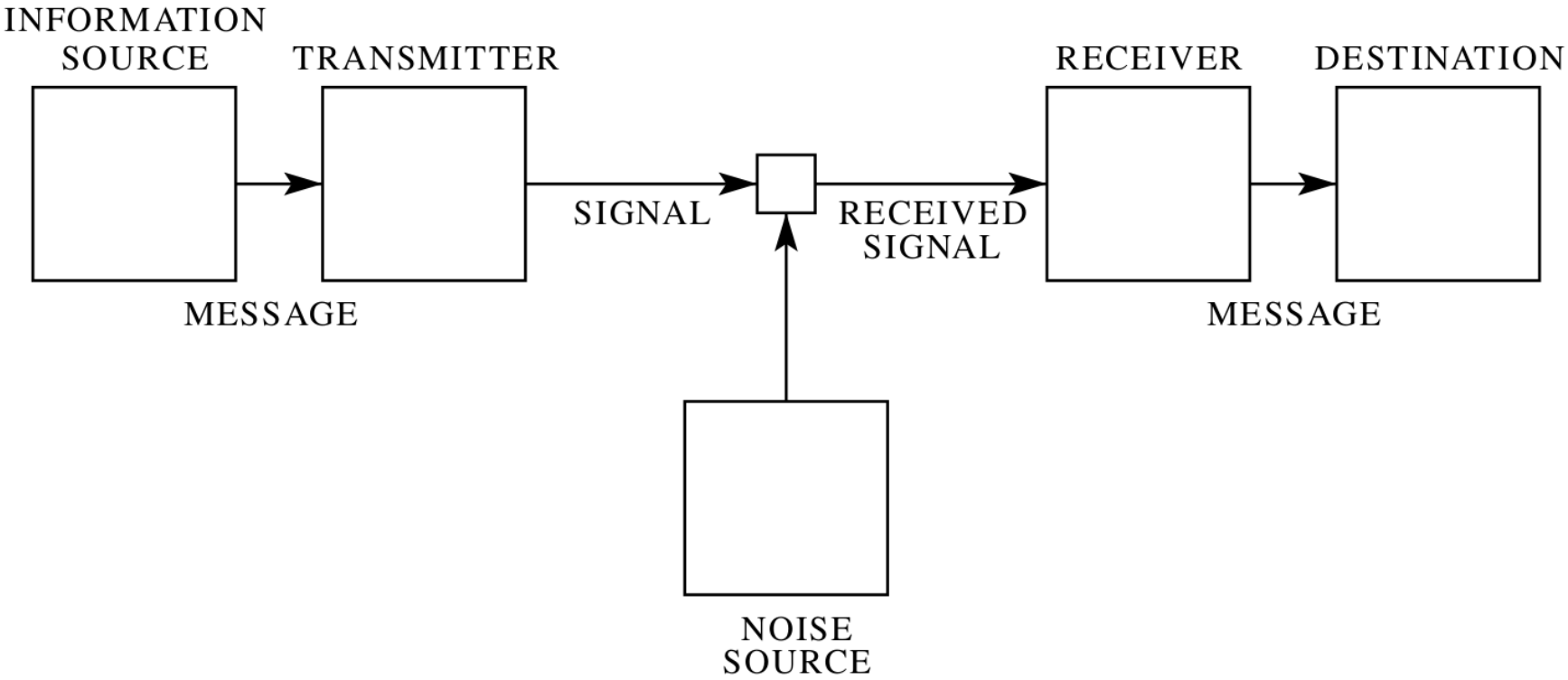
Shannon, Claude Elwood. "A mathematical theory of communication." ACM SIGMOBILE Mobile Computing and Communications Review 5.1 (2001): 3-55.

Setting


INFORMATION SOURCE
TRANSMITTER
RECEIVER
DESTINATION
SIGNAL
RECEIVED SIGNAL
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Capacity and states of a channel

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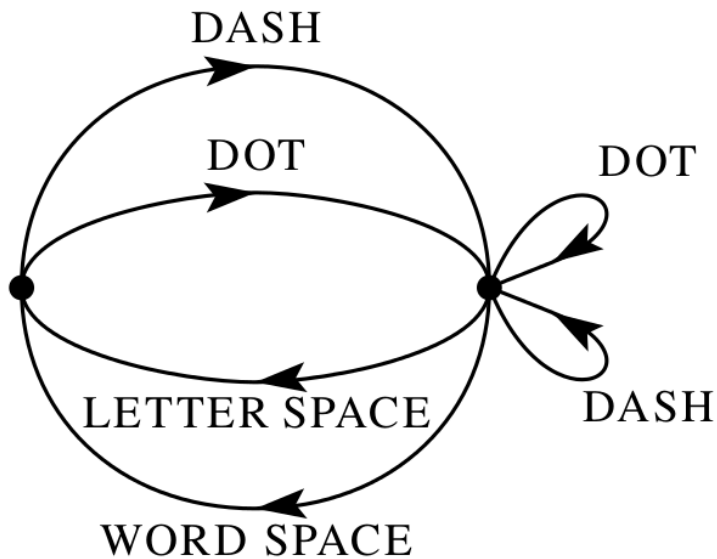

Graphical representation of a Markov process

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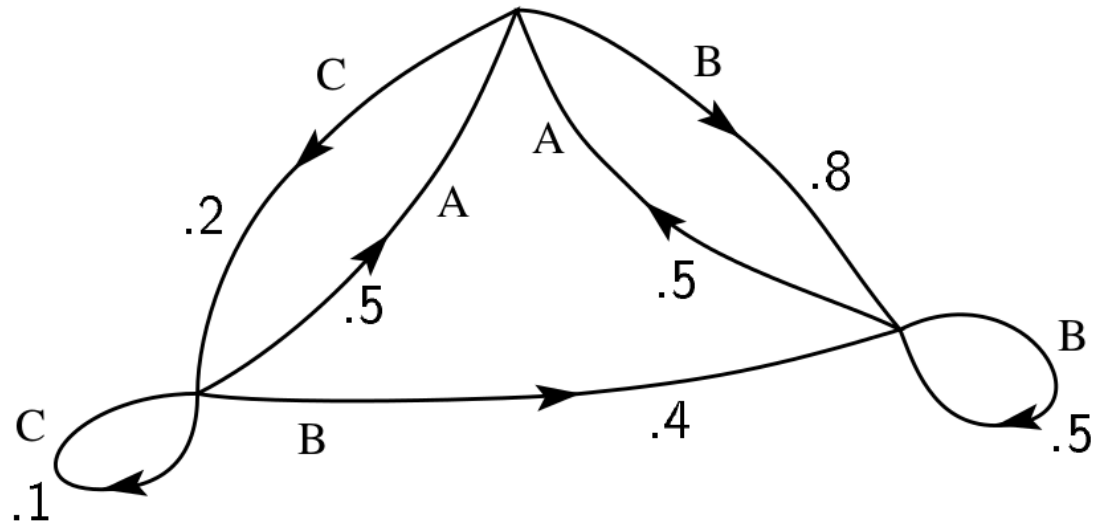

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Using 27 (26+space) alphabet.

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  "One opens a book at random and selects a letter at random on the page. This letter is recorded. The book is then opened to another page and one reads until this letter is encountered. The succeeding letter is then recorded."

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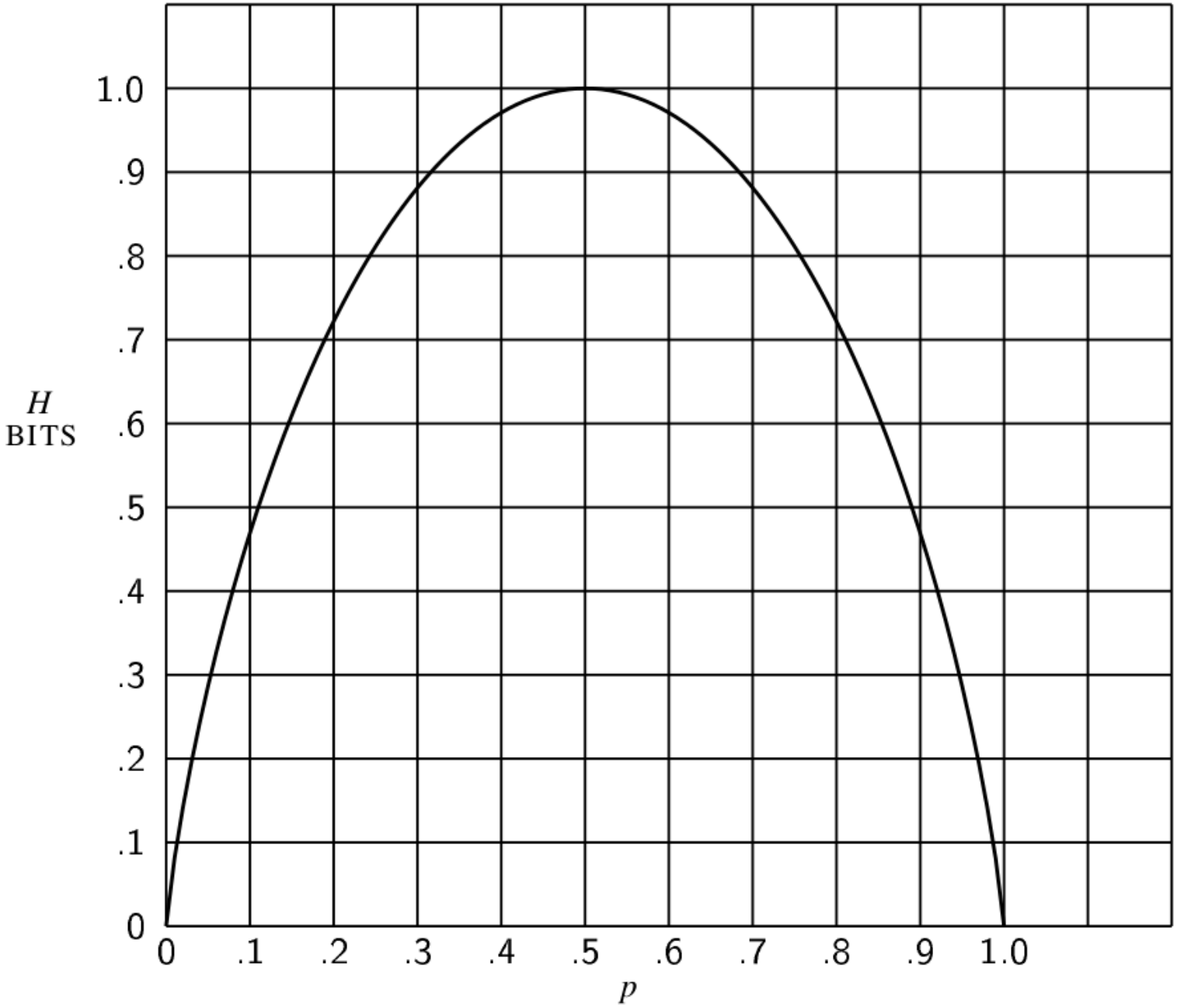
$$H = -(p \log p + q \log q)$$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$x,\ y$ - events

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Entropy of source: bits per symbol

Noiseless case

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Theorem (the fundamental theorem for a noiseless channel)

Let a source have entropy $\boldsymbol{H}$ *bits/symbol and a channel have a capacity* $\boldsymbol{C}$ *bits/second. Then it is possible to encode the output of the source to transmit at the average rate* $\frac{\boldsymbol{C}}{\boldsymbol{H}} - \boldsymbol{\epsilon}$ *symbols/second over the channel where* ϵ *is arbitrarily small.*
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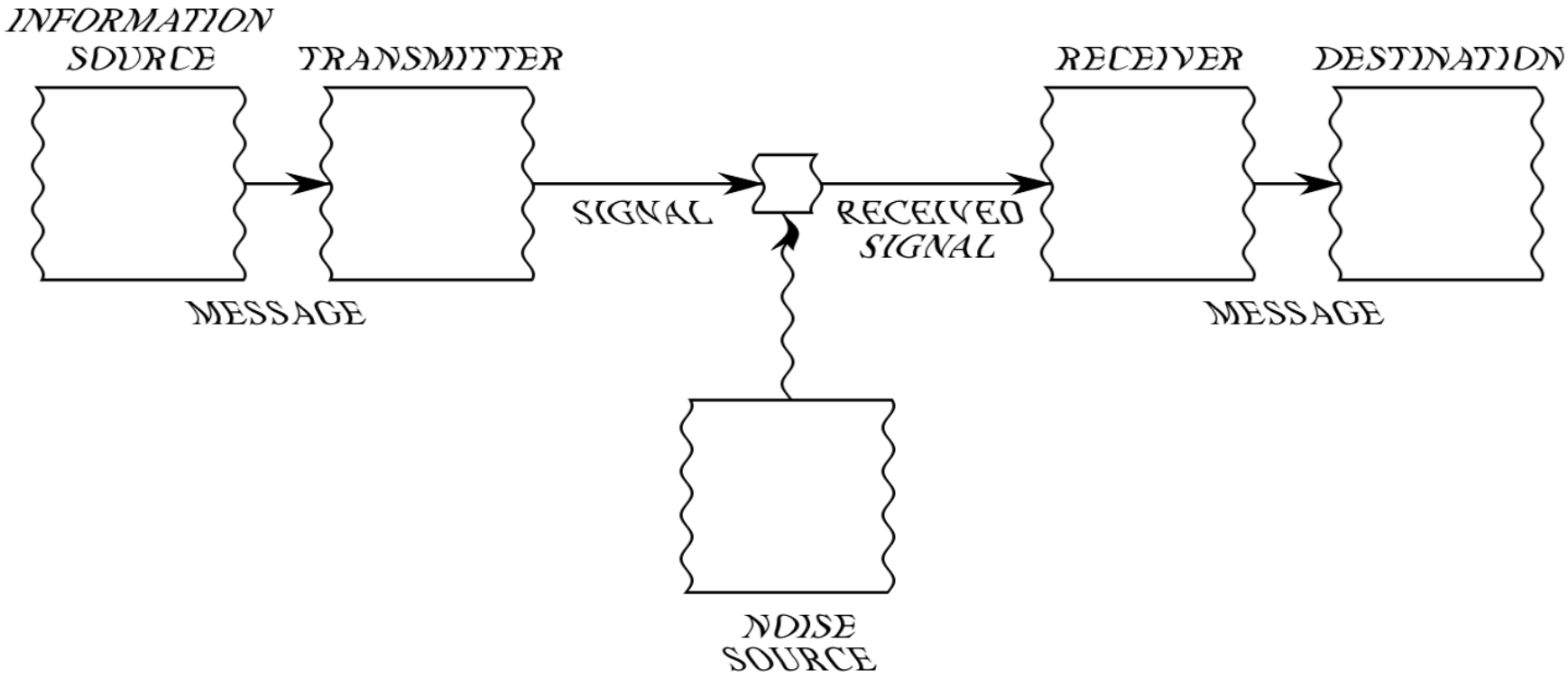
The proof involves constructing an explicit code that achieves the required rate: Shannon-Fano coding.

Noisy case



Source output: x, decoded output: y. Noise: stochastic process as well.

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Thus the actual transmission rate is

$$R = 1000 - 81 = 919 \text{ bits/second}$$

Theorem (the fundamental theorem for a discrete channel with noise)

Let a discrete channel have the capacity $\boldsymbol{C}$ *and a discrete source the entropy per second* $\boldsymbol{H}$*. If* $\boldsymbol{H} \leq \boldsymbol{C}$*, there* **exists** *a coding system with an arbitrarily small frequency of errors (or an arbitrarily small equivocation* $H_y(x)$*) during transmission. If* $\boldsymbol{H} > \boldsymbol{C}$ *it is possible to encode the source so that the equivocation is less than* $\boldsymbol{H} - \boldsymbol{C} + \boldsymbol{\epsilon}$ *where* ϵ *is arbitrarily small. There is no method of encoding which gives an equivocation less than* $H - C$*.*

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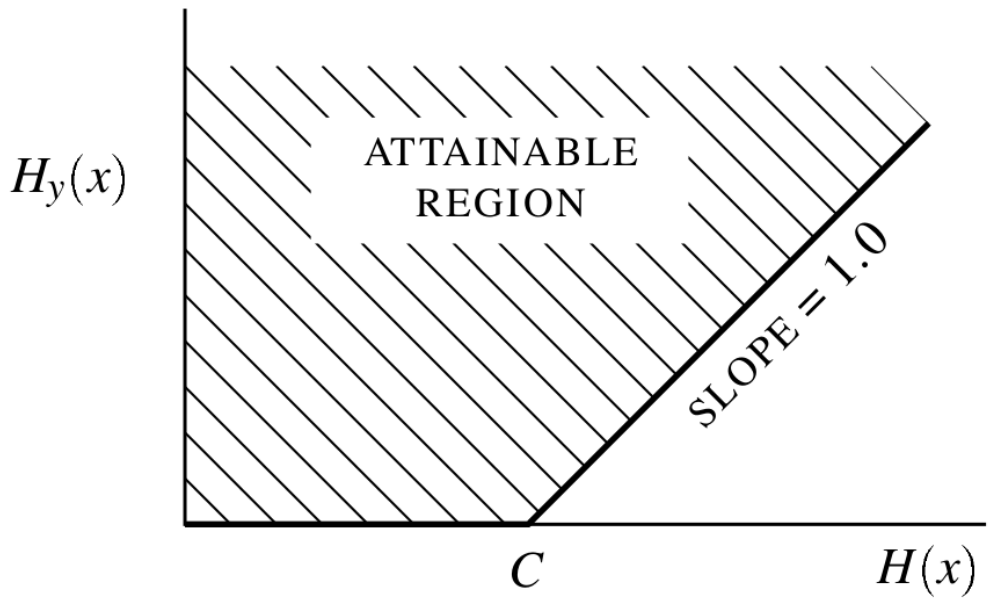


Shannon-Fano coding

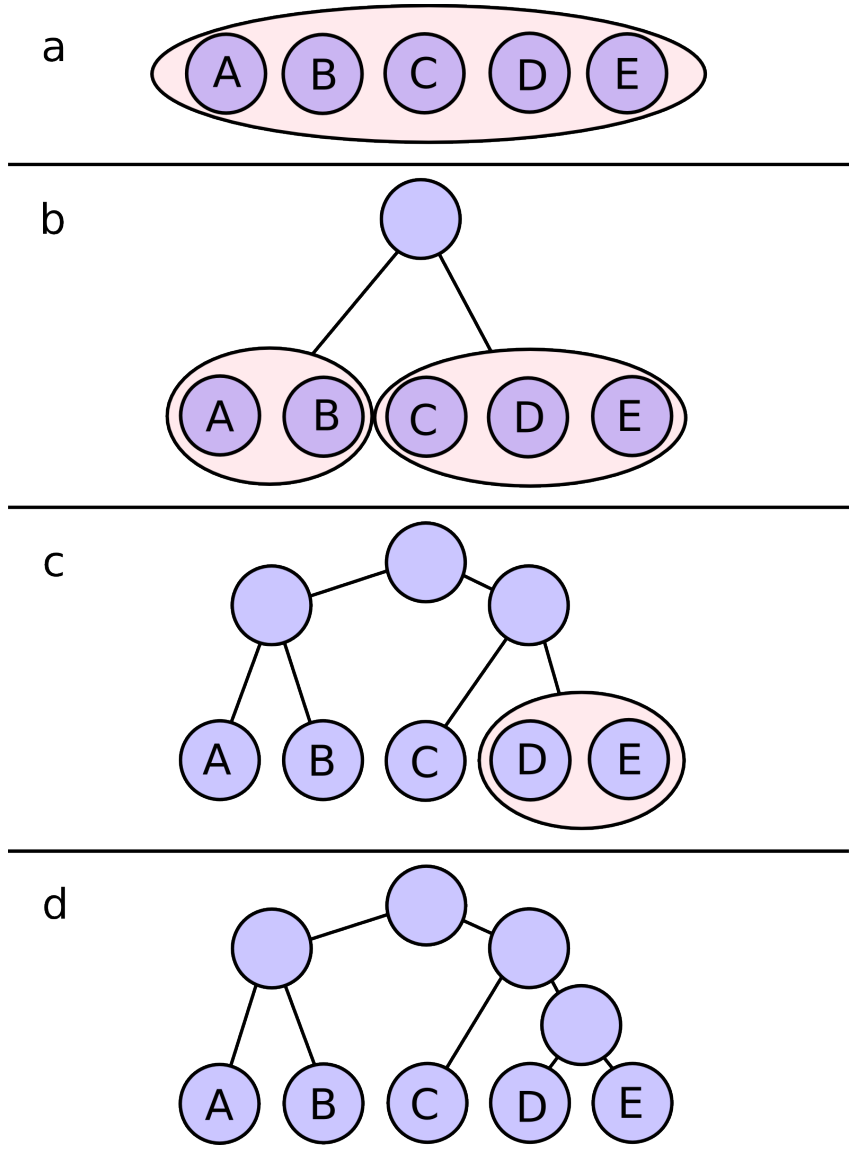
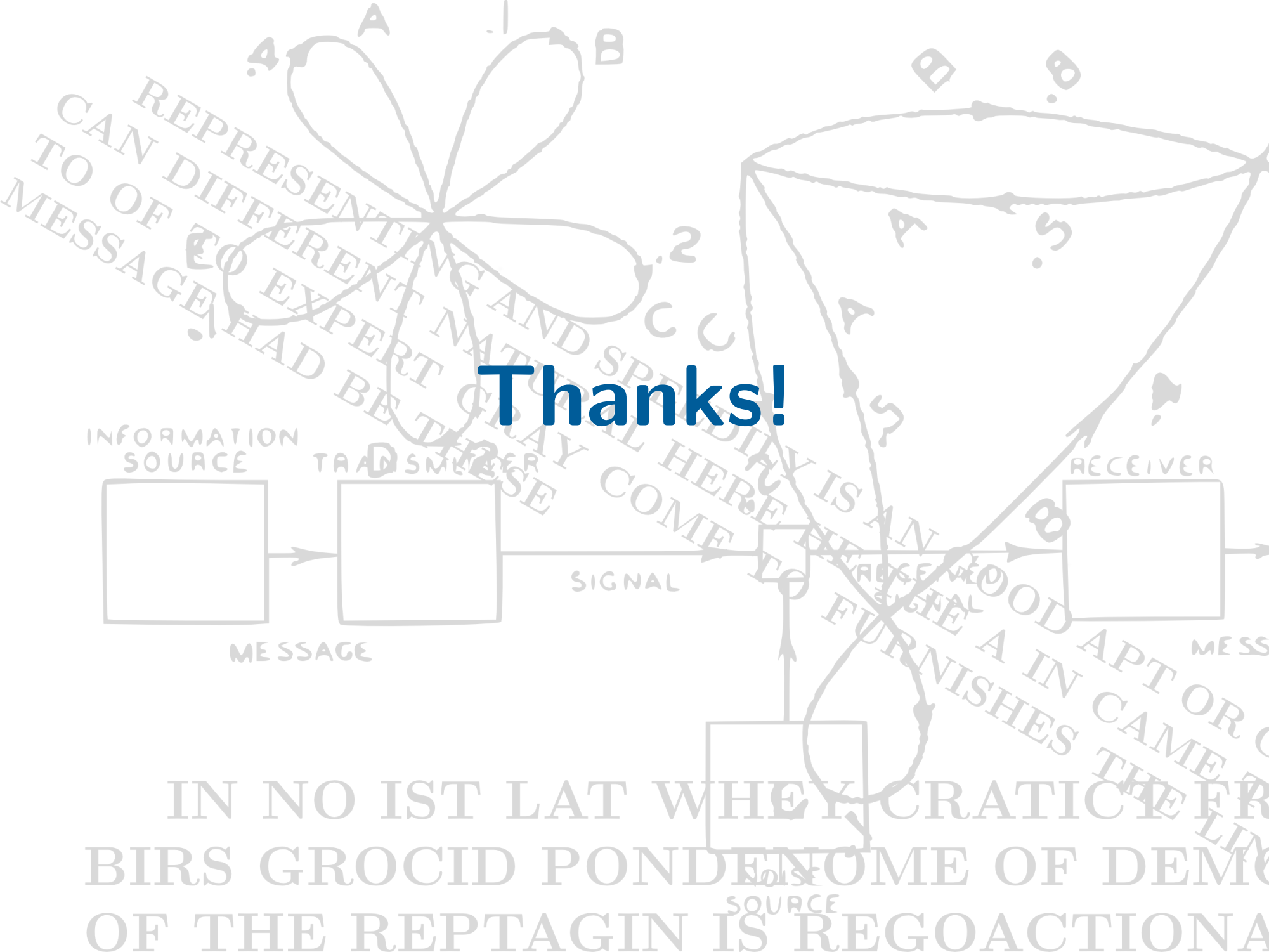


Image: Wikimedia

Thanks!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