REVIEW QUESTIONS FOR TEST 4

Power series

1) Give a definition of the power series. What is the radius of convergence / interval of convergence of a power series? Explain how to compute them for a given series.
2) Determine the radius and interval of convergence for the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 4^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3+1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n(n^3+1)} x^n$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n7^n} x^n$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$

(h) $\displaystyle\sum_{n=1}^{\infty} 3^n (2x-1)^n.$

Representing functions by power series

3) Explain how to differentiate / integrate a power series. What is the impact of these operations on the radius of convergence?
4) Give a power series representations of the following functions:

(a) $f(x) = \dfrac{1}{1-x}$

(b) $f(x) = \dfrac{4}{2x+1}$

(c) $f(x) = \dfrac{x}{1-x}$

(d) $f(x) = \dfrac{x}{3x^2-1}$

(e) $f(x) = \dfrac{1}{x^2+2x+2}$

(f) $f(x) = \dfrac{4x+1}{x^2+6x+10}.$

In the following questions use partial fractions decomposition first, then obtain the expansions of the resulting fractions:

(g) $f(x) = \dfrac{2x-4}{x^2-4x+3}$

(h) $f(x) = \dfrac{2x+3}{x^2+3x+2}.$

(i) $f(x) = \dfrac{3x^2-5x+5}{(x-2)(x^2+3)}$

(j) $f(x) = \dfrac{3x^2+2x+1}{(x+1)(x^2+x+2)}.$

Taylor and Maclaurin series

5) Write down the formula for Taylor series. What needs to be changed to obtain Maclaurin series? What is the expression for $T_n(x)$, the n-th degree Taylor polynomial? What is the Taylor inequality, and how is it used to estimate the error in approximating $f(x)$ with its Taylor polynomial $T_n(x)$? Write down the important Maclaurin series you know.
6) Obtain Maclaurin series for the following functions, using the definition or any other convenient method. Do not show that $R_n(x) \to 0$.

(a) $f(x) = (1-x)^{-2}$
(b) $f(x) = (x+3)^2$
(c) $f(x) = \cos x$
(d) $f(x) = \sin 2x$
(e) $f(x) = \ln(1+x)$
(f) $f(x) = x^2 \ln(1+x)$
(g) $f(x) = e^{x^3}$
(h) $f(x) = \sqrt[4]{(1-x)}$
(i) $f(x) = (2+x)^{-2/3}$
(j) $f(x) = x \cos 3x$
(k) $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ 1/2, & x = 0 \end{cases}$
(l) $f(x) = \dfrac{x^2}{\sqrt{2+x}}$.

7) Compute the Taylor series of the following functions centered at the specified a. Do not show that $R_n(x) \to 0$.

(a) $f(x) = x^3 + 4x^2 + x + 3, \quad a = 2$
(b) $f(x) = \ln x, \quad a = 1$
(c) $f(x) = e^{3x}, \quad a = 2$
(d) $f(x) = \sqrt{x}, \quad a = 9$
(e) $f(x) = \cos x, \quad a = \pi/4$
(f) $f(x) = \sin x, \quad a = \pi/6$.

8) Use series to compute the given limits:

(a) $\displaystyle\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2}$
(b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$
(c) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{1 + x - e^x}$
(d) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5}$.

Applications of Taylor polynomials

9) Explain the method for approximating functions with Taylor polynomials, and its purposes. How to choose the center of such a polynomial? Explain how the singularities of a function influence the radius of convergence if its Taylor polynomial.

10) Find the Taylor polynomial T_3 of degree 3 for the following functions, centered at the given a:

(a) $f(x) = e^x, \quad a = 1$
(b) $f(x) = \sin x, \quad a = \pi/6$
(c) $f(x) = \cos 2x, \quad a = \pi/4$
(d) $f(x) = e^x \sin x, \quad a = 0$
(e) $f(x) = x^2 \ln(1+x), \quad a = 0$
(f) $f(x) = x \sin x, \quad a = 0$
(g) $f(x) = \sqrt{x}, \quad a = 4$
(h) $f(x) = e^{x^2}, \quad a = 0$.

11) For the functions of the previous question, estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $|x - a| \leq 1/2$. Use the formula

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1},$$

which holds when $|f^{(n+1)}(x)| \leq M$.

Answer key.

2) (a) $R = 1$, $[-1, 1)$
(b) $R = 4$, $[1 - 4, 1 + 4]$
(c) $R = 1$, $[2 - 1, 2 + 1]$
(d) $R = +\infty$, $(-\infty, \infty)$
(e) $R = 2$, $[-2, 2]$
(f) $R = 7$, $(-7, 7]$
(g) $R = +\infty$, $(-\infty, \infty)$
(h) $R = 1/6$, $(1/2 - 1/6\,, 1/2 + 1/6)$

4) (a) $f(x) = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$

(b) $f(x) = \sum_{n=0}^{\infty} 4(-2)^n x^n, \qquad |x| < 1/2$

(c) $f(x) = \sum_{n=0}^{\infty} x^{n+1}, \qquad |x| < 1$

(d) $f(x) = \sum_{n=0}^{\infty} -3^n x^{2n+1}, \qquad |x| < 1/\sqrt{3}$

(e) $f(x) = \sum_{n=0}^{\infty} (-1)^n (x+1)^{2n}, \qquad |x+1| < 1$

(f) $f(x) = \sum_{n=0}^{\infty} 4(-1)^n (x+3)^{2n+1} - \sum_{n=0}^{\infty} 11(-1)^n (x+3)^{2n}, \qquad |x+3| < 1$

(g) Taylor series at $a = 2$:

$f(x) = \sum_{n=0}^{\infty} -2(x-2)^{2n+1}, \qquad |x-2| < 1.$

Maclaurin series:

$f(x) = \sum_{n=0}^{\infty} \left(-1 - 3^{-1-n}\right) x^n, \qquad |x| < 1.$

Note that for both series the radii of convergence are 1 because the function has a singularity at $x = 1$, the midpoint between $a = 0$ and $a = 2$.

(h) $f(x) = \sum_{n=0}^{\infty} -8 \cdot 4^n (x + 3/2)^{2n+1}, \qquad |x + 3/2| < 1/2$

(i) $f(x) = \sum_{n=0}^{\infty} \left(\frac{(-1)^{n+1}}{3^{n+1}} - \frac{1}{2^{2n+1}} \right) x^{2n} + \sum_{n=0}^{\infty} \left(\frac{2(-1)^n}{3^{n+1}} - \frac{1}{2^{2n+2}} \right) x^{2n+1}, \qquad |x| < 1/2$

(j) Note that the second factor in denominator can be written as $x^2 + x + 2 = (x + 1/2)^2 + 3/4$, whereas the first is $(x + 1/2) + 1/2$. We can therefore expand the given function about $a = -1/2$; equivalently, in terms of powers of $(x + 1/2)$. The resulting Taylor series has different expressions for the even and odd coefficients:

$$f(x) = \sum_{n=0}^{\infty} \left(2^{2n+1} + 2(-1)^{n+1} \left(\frac{4}{7}\right)^{n+1} \right) (x + 1/2)^{2n}$$
$$+ \sum_{n=0}^{\infty} \left(-2^{2n+2} + 2(-1)^n \left(\frac{4}{7}\right)^{n+1} \right) (x + 1/2)^{2n+1}, \qquad |x + 1/2| < 1/2.$$

6) (a) $f(x) = \sum_{n=0}^{\infty} (n+1)x^n$

(b) $f(x) = x^2 + 6x + 9$

(c) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

(d) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)!}$

(e) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

(f) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{n+1}$

(g) $f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$

(h) $f(x) = \sum_{n=0}^{\infty} (-1)^n \binom{1/4}{n} x^n$

(i) $f(x) = \sum_{n=0}^{\infty} 2^{-n-2/3} \binom{-2/3}{n} x^n$

(j) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n} x^{2n+1}}{(2n)!}$

(k) $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{(2n)!}$

(l) $f(x) = \sum_{n=0}^{\infty} 2^{-n-1/2} \binom{-1/2}{n} x^{n+2}$.

7) (a) $f(x) = 29 + 29(x-2) + 10(x-2)^2 + (x-2)^3$

(b) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$

(c) $f(x) = \sum_{n=0}^{\infty} \frac{e^6 3^n (x-2)^n}{n!}$

(d) $f(x) = \sum_{n=0}^{\infty} 3^{1-2n} \binom{1/2}{n} (x-9)^n$

(e) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{2}} \frac{(x-\pi/4)^{2n}}{(2n)!} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{2}} \frac{(x-\pi/4)^{2n+1}}{(2n+1)!}$

(f) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{2}} \frac{(x-\pi/6)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{3}{2}} \frac{(x-\pi/6)^{2n+1}}{(2n+1)!}$.

8) (a) $1/2$

(b) $-1/8$

(c) -1

(d) $1/120$.

10) (a) $T_3(x) = e + e(x-1) + \frac{e(x-2)^2}{2} + \frac{e(x-1)^3}{6}$

(b) $T_3(x) = \frac{1}{2} + \frac{\sqrt{3}\,\left(x-\frac{\pi}{6}\right)}{2} - \frac{\left(x-\frac{\pi}{6}\right)^2}{4} - \frac{\sqrt{3}\,\left(x-\frac{\pi}{6}\right)^3}{12}$

(c) $T_3(x) = -2\left(x-\frac{\pi}{4}\right) + \frac{4\left(x-\frac{\pi}{4}\right)^3}{3}$

(d) $T_3(x) = x + x^2 + \frac{x^3}{3}$

(e) $T_3(x) = x^3$

(f) $T_3(x) = x^2$

(g) $T_3(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}$

(h) $T_3(x) = 1 + x^2$.

11) In what follows we estimate the remainder $R_n(x) = f(x) - T_n(x)$. We used the assumption that $|x-a| \le 1/2$.

(a) $|R_3(x)| \le \dfrac{e^{3/2}}{4!}\left(\dfrac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |e^x| \le e^{3/2}$ for $|x-1| \le 1/2$.

(b) $|R_3(x)| \le \dfrac{1}{4!}\left(\dfrac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |\sin(x)| \le 1$ for $|x-\pi/6| \le 1/2$.

(c) $|R_3(x)| \le \dfrac{2^4}{4!}\left(\dfrac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |2^4\cos(x)| \le 2^4$ for $|x-\pi/4| \le 1/2$.

(d) $|R_3(x)| \le \dfrac{4e^{1/2}}{4!}\left(\dfrac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |-4e^x\sin x| \le 4e^{1/2}$ for $|x| \le 1/2$.

(e) $|R_3(x)| \le \dfrac{136}{4!}\left(\dfrac{1}{2}\right)^4$ in view of

$$\begin{aligned}
|f^{(4)}(x)| &= \left|-\frac{6x^2}{(x+1)^4} + \frac{16x}{(x+1)^3} - \frac{12}{(x+1)^2}\right| \\
&\le \left|\frac{6x^2}{(x+1)^4}\right| + \left|\frac{16x}{(x+1)^3}\right| + \left|\frac{12}{(x+1)^2}\right| \\
&\le \left|\frac{6\cdot(1/2)^2}{(-1/2+1)^4}\right| + \left|\frac{16\cdot 1/2}{(-1/2+1)^3}\right| + \left|\frac{12}{(-1/2+1)^2}\right| \\
&= 6\cdot 4 + 16\cdot 4 + 12\cdot 4 = 136 \qquad \text{for } |x| \le 1/2.
\end{aligned}$$

Here the absolute values of the fractions in the second line are estimated as the largest value in the numerator over the smallest value in the denominator.

(f) $R_3(x) \le \dfrac{9/2}{4!}\left(\dfrac{1}{2}\right)^4$ in view of

$$\begin{aligned}
|f^{(4)}(x)| &= |x\sin(x) - 4\cos(x)| \\
&\le |x\sin(x)| + 4|\cos(x)| \\
&\le \frac{1}{2}\cdot 1 + 4 = 9/2 \qquad \text{for } |x| \le 1/2.
\end{aligned}$$

(g) $R_3(x) \le \dfrac{15}{343\sqrt{14}}\cdot\dfrac{1}{4!}\left(\dfrac{1}{2}\right)^4$ in view of

$$|f^{(4)}(x)| = \left|-\frac{15}{16x^{7/2}}\right| \le \frac{15}{343\sqrt{14}} \qquad \text{for } |x-4| \le 1/2.$$

We used here that the absolute value of $f^{(4)}$ is decreasing, and so the maximum is attained at the left endpoint of $|x-4| \le 1/2$.

(h) $R_3(x) \le \dfrac{25e^{1/4}}{4!}\left(\dfrac{1}{2}\right)^4$ in view of

$$\begin{aligned}
|f^{(4)}(x)| &= \left|48e^{x^2}x^2 + 12e^{x^2} + 16e^{x^2}x^4\right| \\
&\le 48e^{1/2}(1/2)^2 + 12e^{(1/2)^2} + 16e^{(1/2)^2}(1/2)^4 \\
&= 25e^{1/4} \qquad \text{for } |x| \le 1/2.
\end{aligned}$$