REVIEW QUESTIONS FOR TEST 2

Arc length

1) Write down the arc length formula, for integration both in terms of x and y. How are the limits of integration determined in it? Which expression can be interpreted as the element of arc length ds?
2) Find the length of the following curves:
   (a) $36y^2 = (x^2 - 4)^3, \quad 2 \le x \le 3, \quad y \ge 0$
   (b) $x = \dfrac{y^5}{10} + \dfrac{1}{6y^3}, \quad 1 \le y \le 3$
   (c) $y = \ln(\cos x), \quad 0 \le x \le \pi/6$
   (d) $y = \dfrac{1}{2}(e^x + e^{-x}), \quad 0 \le x \le 1$
   (e) $y = \sqrt{x - x^2} + \arcsin(\sqrt{x}), \quad 0 \le x \le 1$
   (f) $x = \ln(1 - y^2), \quad 0 \le y \le \dfrac{1}{2}.$

Areas of surfaces of revolution

3) Write down the formulas for the area of a surface of revolution. What changes when the axis of rotation is the y-axis? Write down the formula for ds (element of arc length) for integration in x. The same for integration in y.
4) Find area of the surface obtained by rotating the following curves about the x-axis:
   (a) $y = \cos(3x), \quad 0 \le x \le \pi/6$
   (b) $x = \dfrac{1}{3}(y^2 + 2)^{3/2}, \quad 1 \le y \le 2$
   (c) $y = \sqrt{1 + e^x}, \quad 0 \le x \le 1.$
5) Find area of the surface obtained by rotating the following curves about the y-axis:
   (a) $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x, \quad 1 \le x \le 2$
   (b) $x^{2/3} + y^{2/3} = 1, \quad 0 \le y \le 1$
   (c) $y = \dfrac{1}{3}x^{3/2}, \quad 0 \le x \le 6.$

Applications to physics

6) Write down the formula for hydrostatic pressure at the depth d. Write down formulas for the coordinates of a centroid of a flat region confined between two curves; assume that the density is constant. What changes when the region lies between a curve and the x-axis? How are the limits of integration determined in the relevant integrals? What can be said about the location of a centroid of a convex shape?
7) See # 6, 15, 10, 15 in Section 8.3.
8) Find centroids of the regions bounded by the given curves:
   (a) $y = \sin x, \ y = 0, \ 0 \le x \le \pi$
   (b) $y = 2 - x^2, \ y = x^2$
   (c) $y = \sin x, \ y = \cos x, \ x = 0, \ x = \pi/4$

(d) $x + y = 2,\ x = y^2$.

Separable differential equations

9) Explain what is a separable equation. Write down the general algorithm that should be applied to separable differential equations. In which case is it possible to express y as an explicit function of x, assuming the equation contains dy/dx? What determines the value of the constant of integration in a particular solution?

What is an orthogonal trajectory? Why are orthogonal trajectories found using a differential equation? Write down the algorithm for finding the equation of orthogonal trajectories to a given family of curves.

10) Solve the differential equations:
(a) $y' = x\sqrt{y}$
(b) $y' + xe^x = 0$
(c) $\dfrac{d\theta}{dt} = \dfrac{t \sec\theta}{\theta e^{t^2}}$
(d) $\dfrac{dz}{dt} + e^{t+z} = 0$.

11) Solve the differential equations and find solutions satisfying the given initial condition:
(a) $y' = 2x\sqrt{1-y^2}, \quad y(0) = 0$
(b) $\dfrac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$
(c) $x \ln x = y(1 + \sqrt{3+y^2})y', \quad y(1) = 1$.

12) Find orthogonal trajectories of the given family of curves, indexed by the constant k:
(a) $x^2 + 3y^2 = k^2$
(b) $y = \dfrac{k}{x}$.

13) See # 46, 47 in Section 9.3.

Models for population growth

14) What equations modelling the population growth do you know? Write them down; are all of them separable? What information can be inferred from the direction field of a differential equation? How does the second factor in the logistic equation ensure that the population never becomes larger than the carrying capacity M?

15) Solve the initial value problem for the logistic equation:

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{2000}\right), \qquad P(0) = 50.$$

(a) Find $P(20)$.
(b) When does the population reach $P(t) = 1600$?

Parametric curves

16) What is a parametric curve? How and when can one obtain the Cartesian equation of a parametric curve? How to compute the slope of the tangent to a parametric curve? Write down the expressions for the area under a parametric curve, and the arc length of a parametric curve. What is the equation for the area of the surface obtained by rotation of a parametric curve?

17) Eliminate the parameter from the given parametric curve and sketch its graph. Show the direction of motion as the parameter increases.
(a) $x = \sin t,\ y = \csc t, \quad 0 < t < \pi/2$

(b) $x = e^t,\ y = e^{-2t}, \quad -\infty < t < \infty.$

18) Find the equation of the tangents to the following curves for the given value of the parameter t or at the given point (x, y):
(a) $x = t^3 + 1,\ y = t^4 + t, \quad t = -1$
(b) $x = t\cos t,\ y = t\sin t, \quad t = \pi$
(c) $x = 1 + \ln t,\ y = t^2 + 2, \quad (1, 3)$
(d) $x = t^2 - t,\ y = t^2 + t + 1, \quad (0, 3).$

19) Find the length of the given curve:
(a) $x = 1 + 3t^2,\ y = 4 + 2t^3, \quad 0 \le t \le 1$
(b) $x = e^t - t,\ y = 4e^{t/2}, \quad 0 \le t \le 2$
(c) $x = t\sin t,\ y = t\cos t, \quad 0 \le t \le 1.$

20) Find the area of surfaces obtained by rotating the following curves about the x-axis:
(a) $x = t^3,\ y = t^2, \quad 0 \le t \le 1$
(b) $x = 2\cos^3 t,\ y = 2\sin^3 t, \quad 0 \le t \le \pi/2.$

21) Find the area of surfaces obtained by rotating the following curves about the y-axis:
(a) $x = 8\sqrt{t},\ y = 2t^2 + 1/t, \quad 1 \le t \le 3$
(b) $x = e^t - t,\ y = 4e^{t/2}, \quad 0 \le t \le 1.$