# SOME IMPORTANT MACLAURIN SERIES AND A COMMENT ON BINOMIAL COEFFICIENTS

The **Maclaurin series** of a function $f$ is the power series expansion of $f(x)$ about 0 in the form

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

It is a special case of Taylor series with $a = 0$. As usual, we assume that $0! = 1$.

Several important Maclaurin series are given below. **You need to know them for Test 4**.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \qquad R = +\infty$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \dots \qquad R = +\infty$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \dots \qquad R = +\infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 \dots \qquad R = 1$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \dots \qquad R = 1$$

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \qquad R = 1$$

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots \qquad R = 1$$

Observe that the series for $\ln(1+x)$ and $\arctan x$ can be obtained from the series for $1/(1+x)$ and $1/(1+x^2)$ respectively, by integration. Binomial coefficients are given by the formula

$$\binom{p}{k} = \frac{p(p-1)(p-2)\dots(p-k+1)}{k!},$$

where $p$ is any real number; $k$ is a positive integer.

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**Binomial coefficients**.

To begin, let's write out what different powers of $(a+b)^p$ expand to, for integer $p = 0, 1, 2, 3, \dots$, like so:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

For larger positive integers $p = 4, 5, 6, \dots$, we could similarly expand $(a+b)^p$. The expression for $(a-b)^p$ follows from $(a+b)^p$ by replacing $b$ with $-b$ and keeping track of the sign of

$(-b)^k = (-1)^k b^k$. The coefficients next to $a^l b^k$ in the expansion on the right-hand side can be written out in the so-called *Pascal's triangle*:

$$\begin{array}{lccccccccccccccc}
p=0: & & & & & & & & 1 & & & & & & & \\
p=1: & & & & & & & 1 & & 1 & & & & & & \\
p=2: & & & & & & 1 & & 2 & & 1 & & & & & \\
p=3: & & & & & 1 & & 3 & & 3 & & 1 & & & & \\
p=4: & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
p=5: & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
p=6: & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
p=7: & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

$$\dots$$

Turns out that the $k$-th number, with $k$ running from 0 to $p$, in the $p$-th line above is given by what we called the binomial coefficient:

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} = \frac{p(p-1)(p-2)\dots(p-k+2)(p-k+1)}{k!}.$$

It is also called *$p$ choose $k$*, because it is the number of ways of choosing $k$ objects among $p$ different objects (think about choosing $k$ marbles among $p$ different marbles numbered $1, 2, \dots, p$).

Let's take the third line above for example, where $p = 3$. The number of ways to choose 0 objects among 3 is **1** — choosing 0 objects means choosing nothing, and you have only *one way* of doing that. That's why this row starts with 1. Next, the number of ways to choose 1 object among 3 objects is **3**: you can choose the first, or the second, or the third one. Then, there is **3** ways to choose 2 objects among 3 objects — you can take *all but* the first, or *all but* the second, or *all but* the third. Finally, there is only **1** way to choose 3 objects among 3 objects — you have to take all of them. So, there we have the row for $p = 3$:

$$1 \quad 3 \quad 3 \quad 1.$$

The rest of the rows of the above table can be given meaning in the same way. And, there is a neat way to compute the numbers in the table: each number is the sum of the two from the previous row, sitting diagonally to the left and to the right of it, like so:

$$\dots$$

$$\begin{array}{lccccccccccccc}
p=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
 & & \swarrow\searrow & & \swarrow\searrow & & \swarrow\searrow & & \swarrow\searrow & & \swarrow\searrow & & \swarrow\searrow & \\
p=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

$$\dots$$

Even though it is called *Pascal's triangle*, it (as is the case with many discoveries) was known long before being found by the person it was named after.