THE VEECH 2-CIRCLE PROBLEM AND NON-INTEGRABLE FLAT DYNAMICAL SYSTEMS

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Point Distributons Webinar
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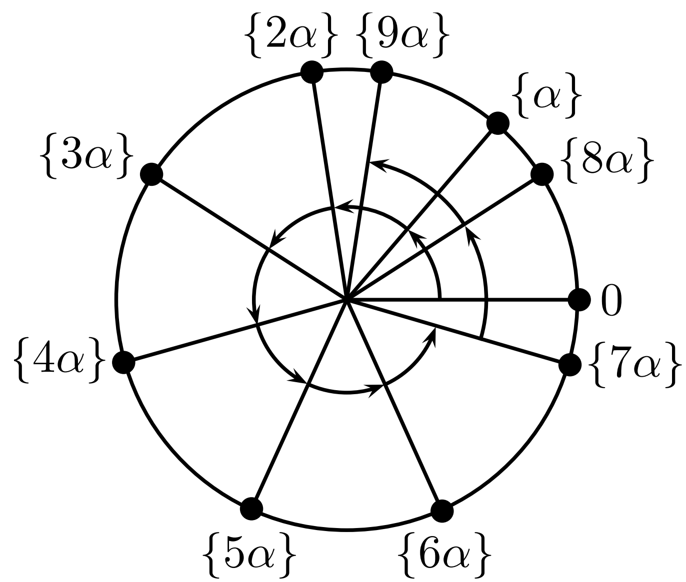

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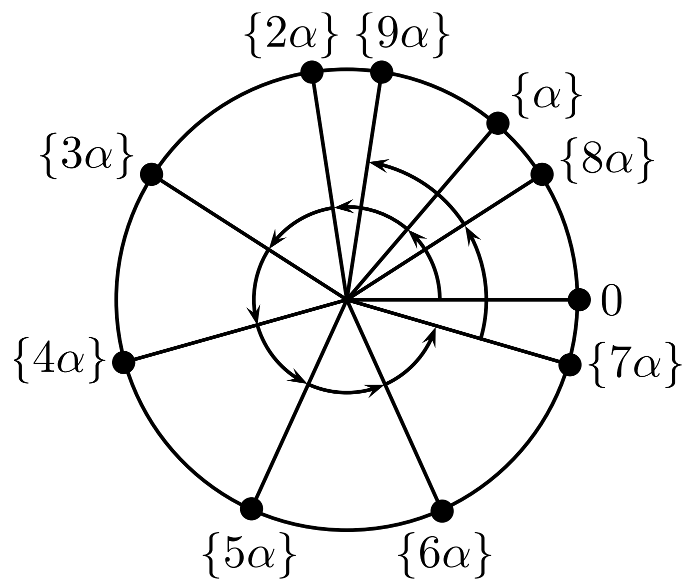

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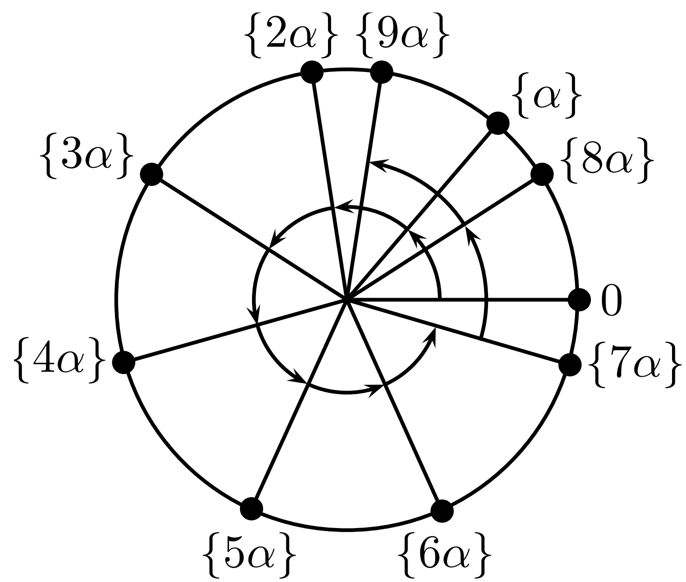

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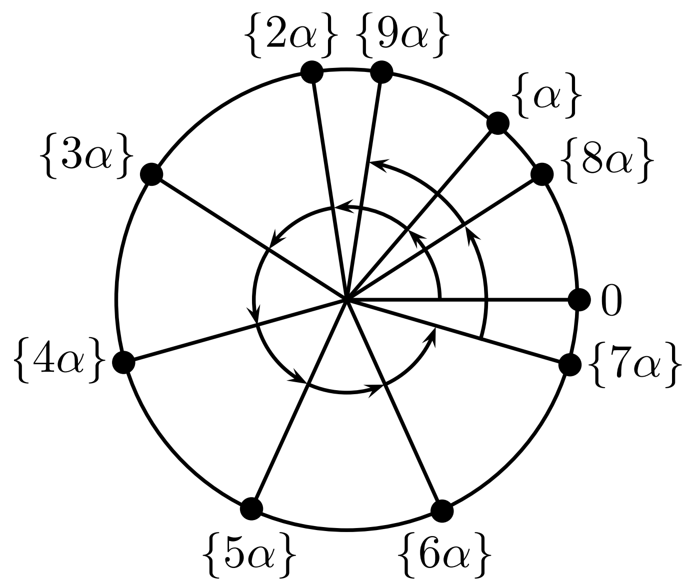

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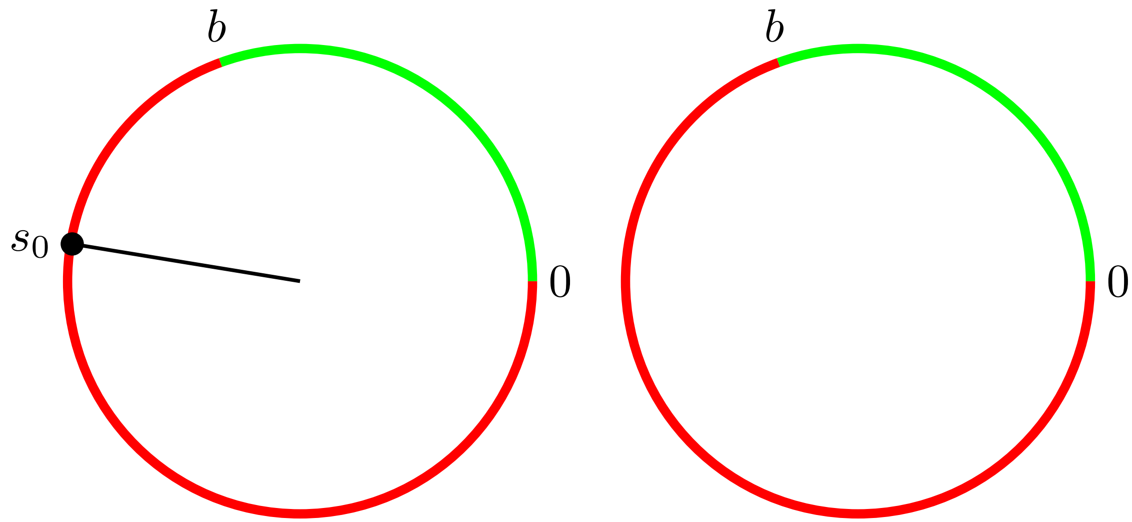
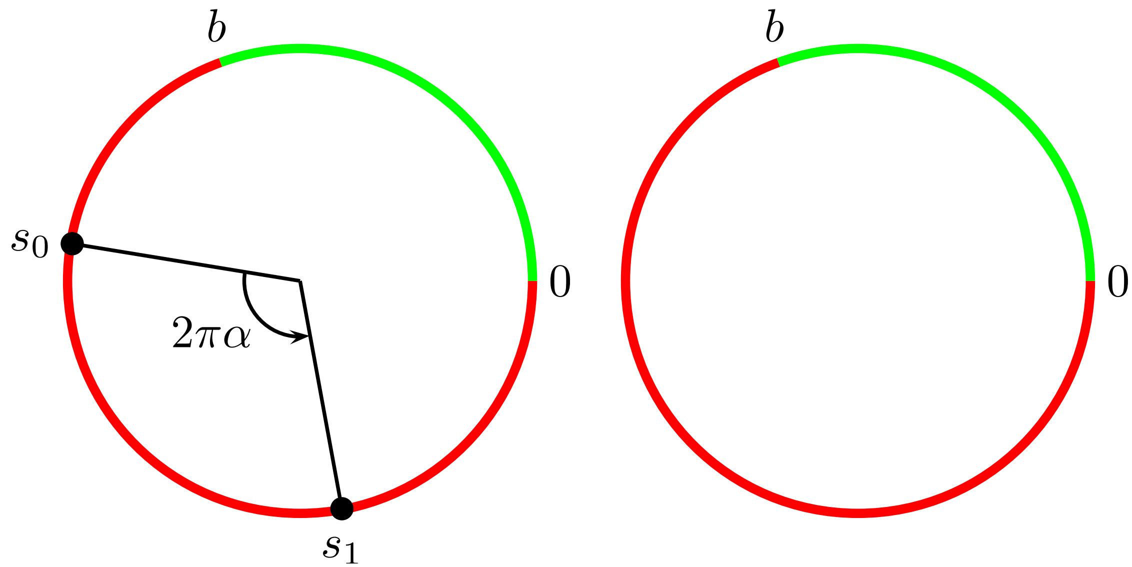
uniform-periodic dichotomy


b
s_0
0
b
0


b
s_0
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$2\pi\alpha$
s_1
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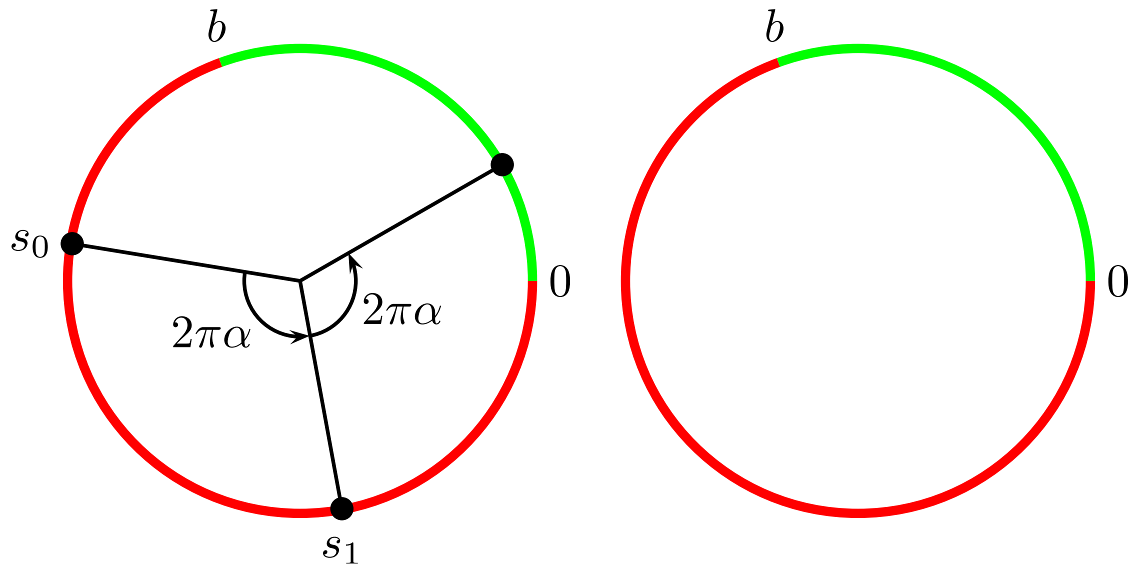
b
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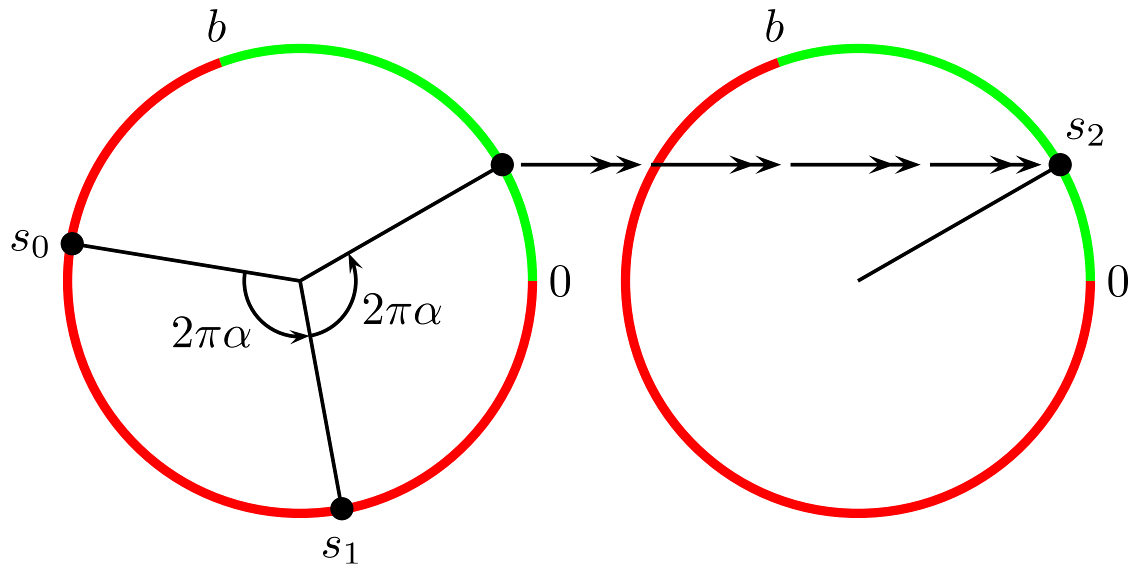
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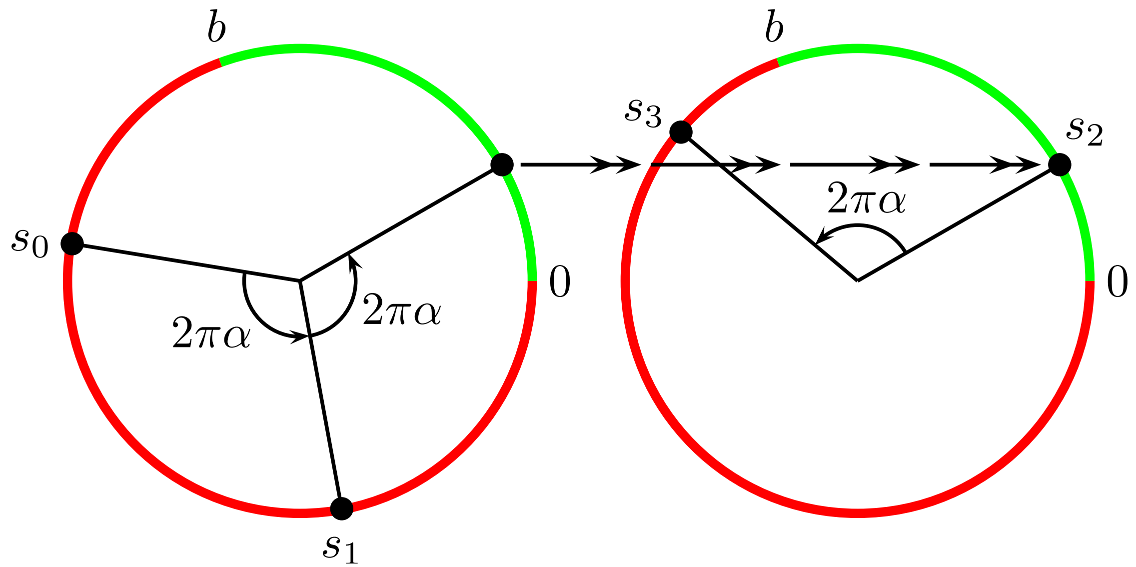
$2\pi\alpha$
s_1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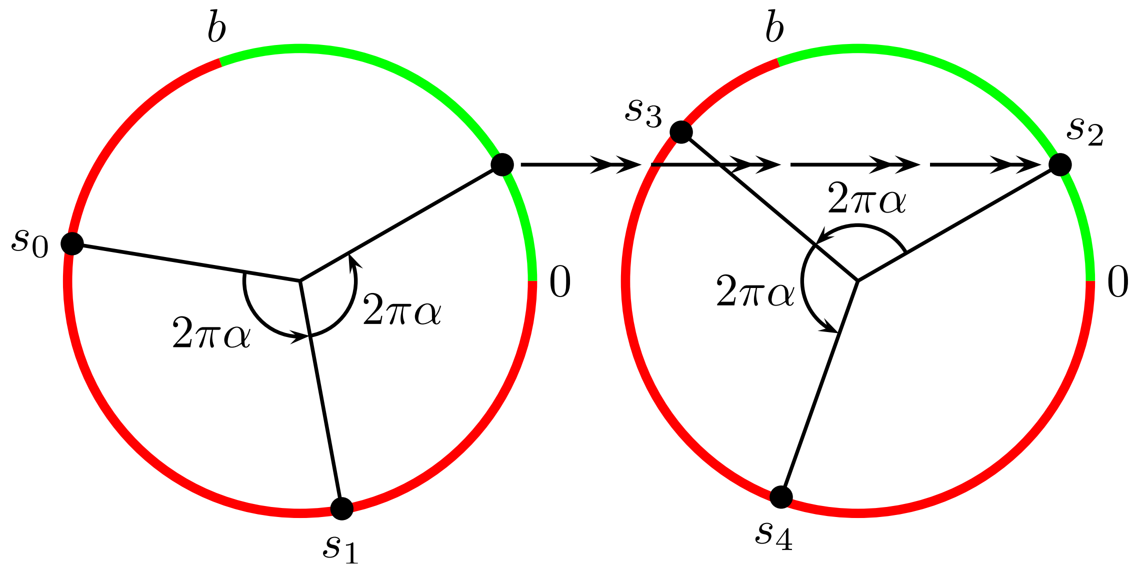
b
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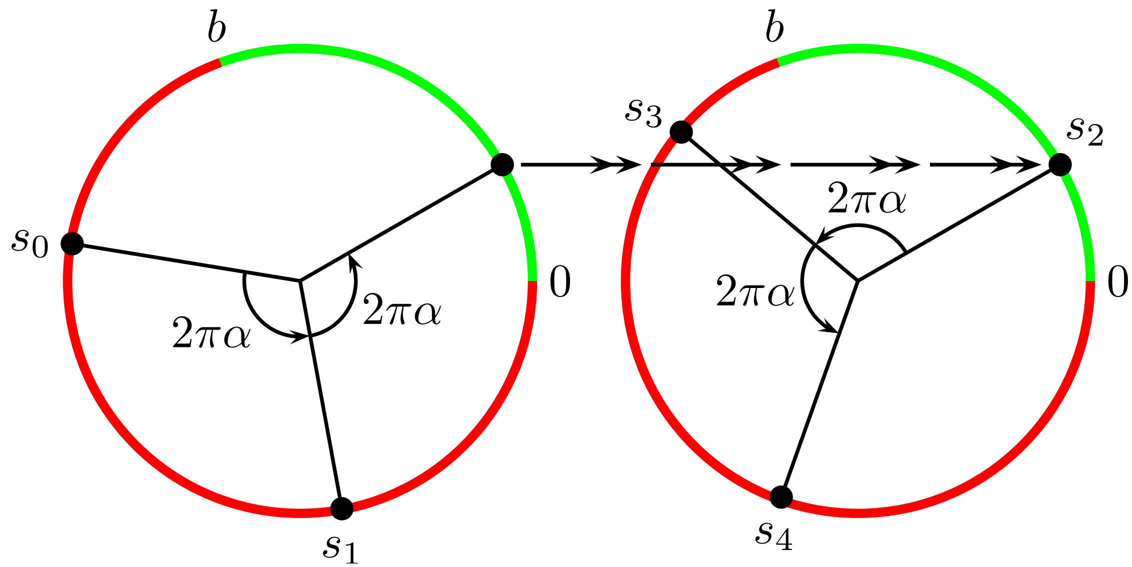
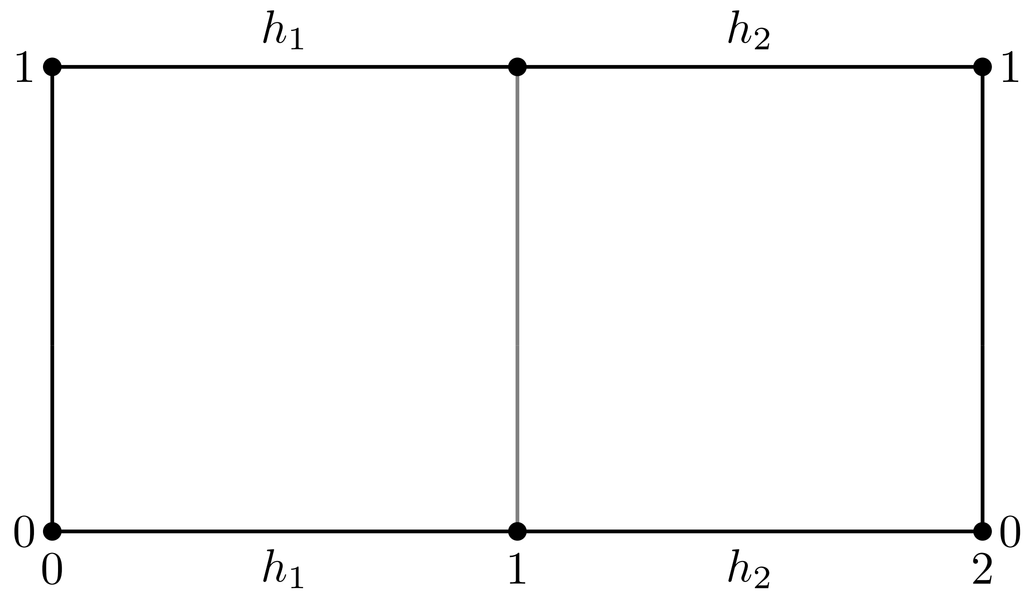
$2\pi\alpha$
$2\pi\alpha$
0
s_1
s_4



assume that α is irrational

when do we have equidistribution ?

viewed in terms of geodesic flow of slope α on a flat surface



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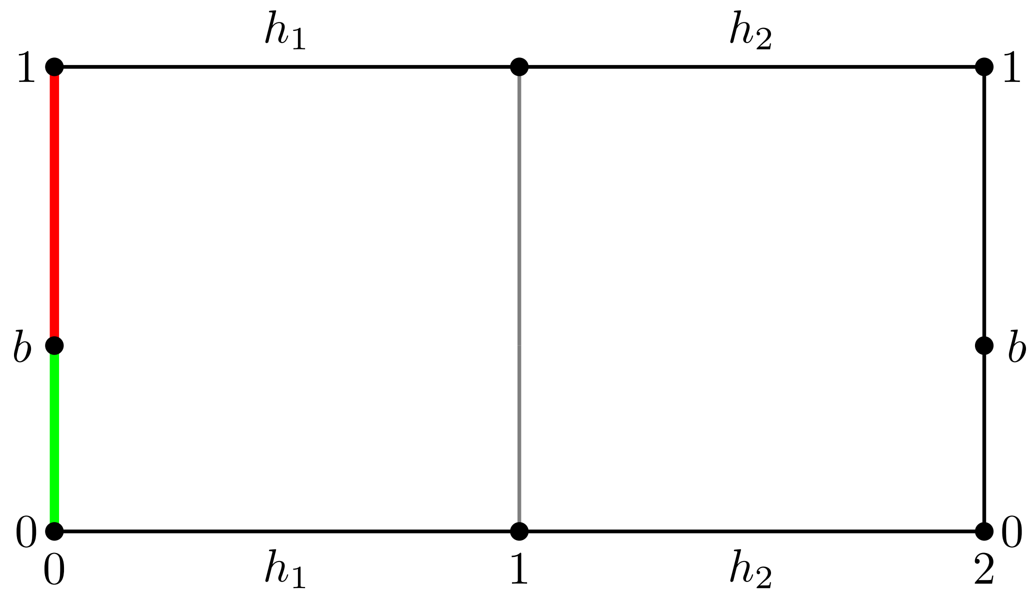

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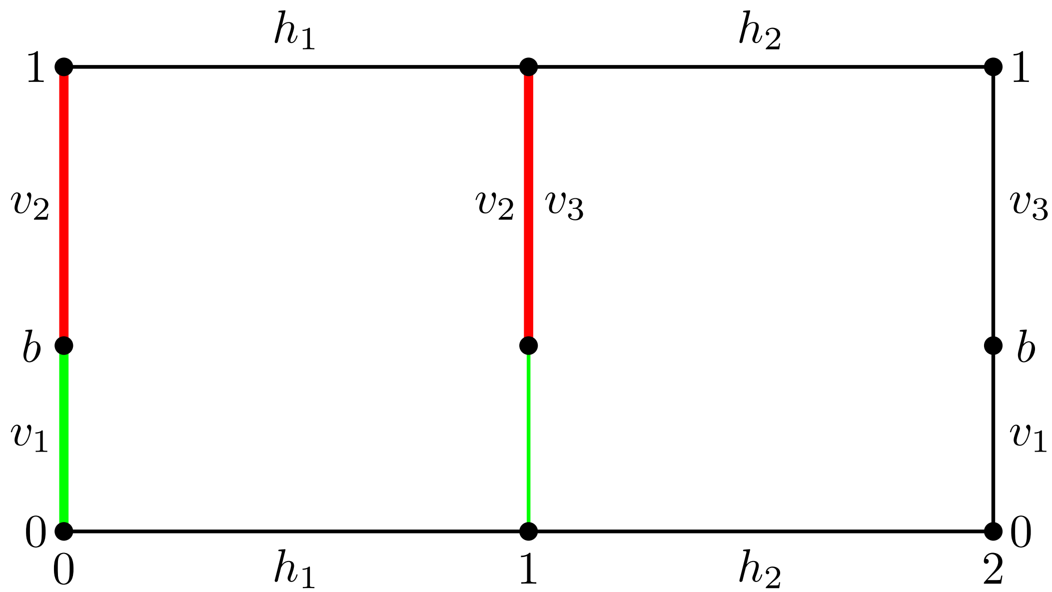

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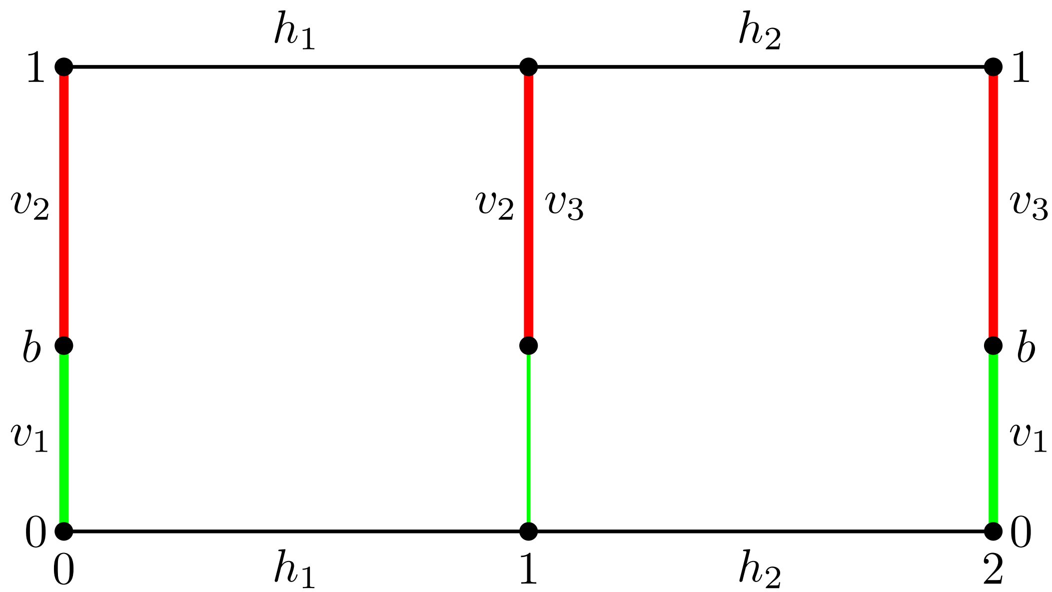

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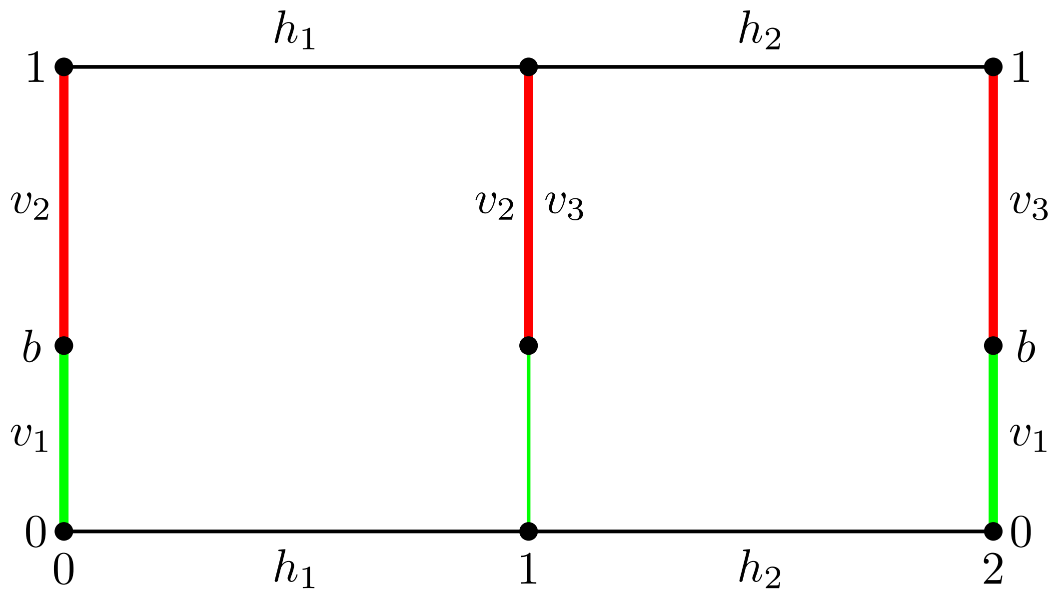
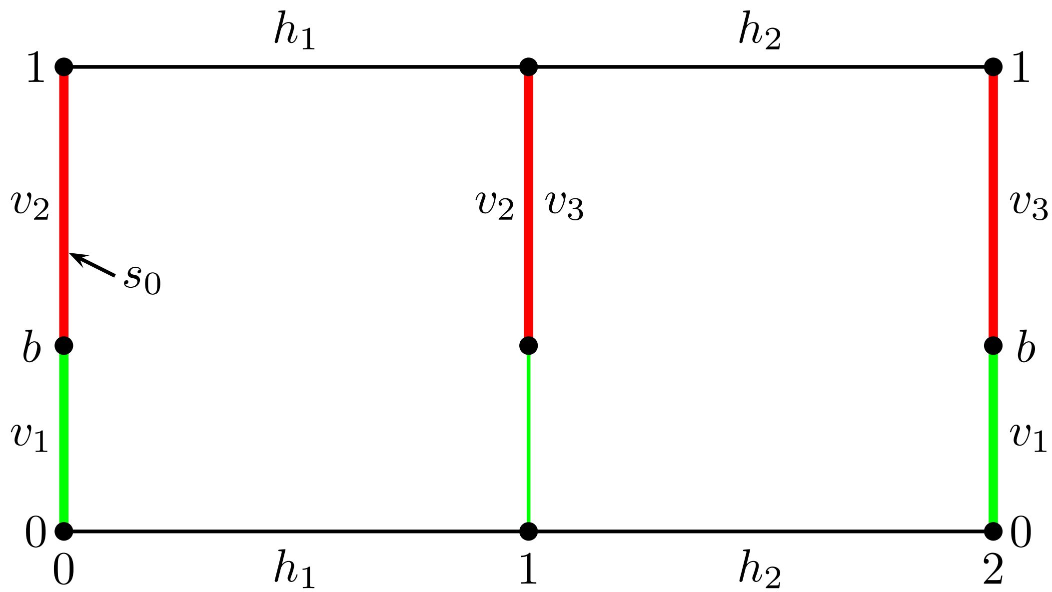


2-square-b surface

viewed in terms of geodesic flow of slope α on a flat surface



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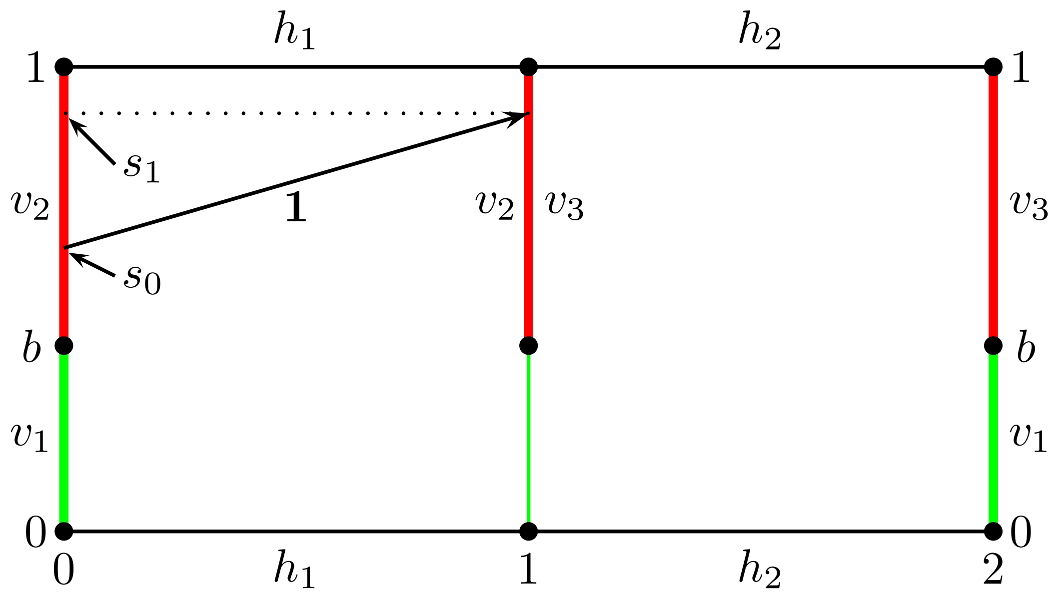

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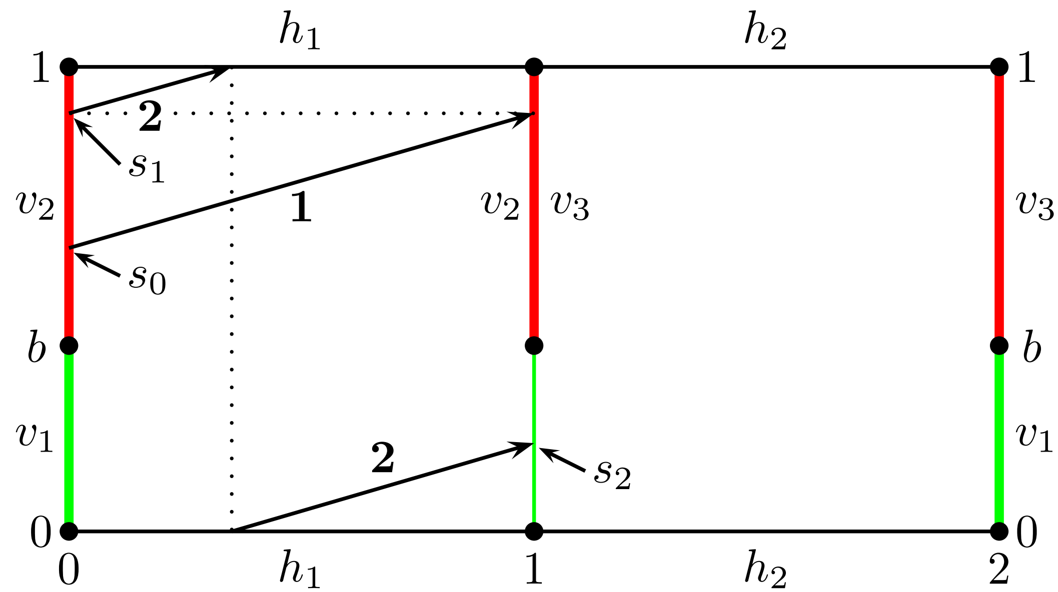

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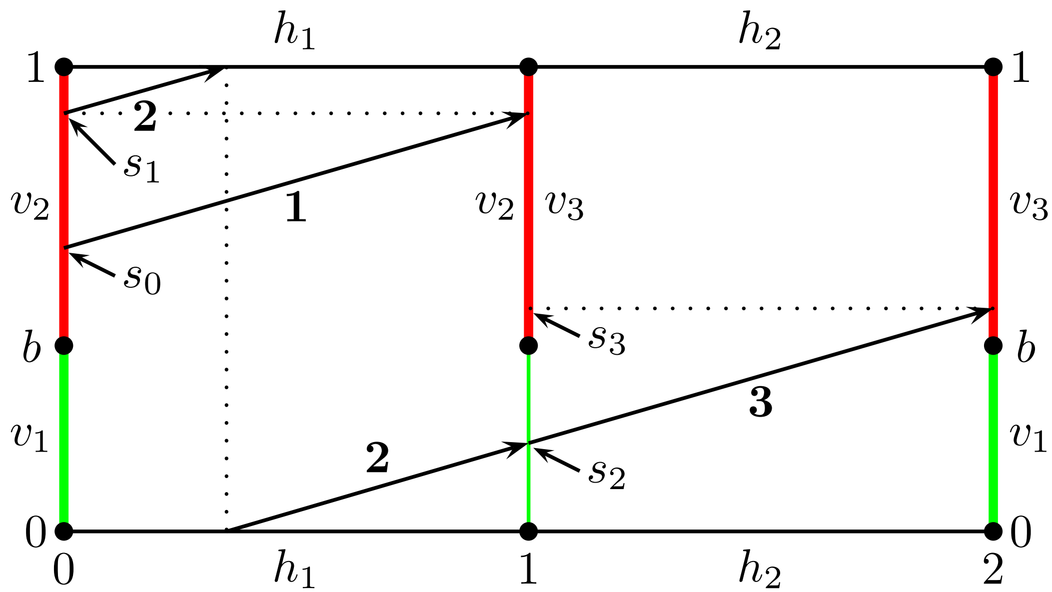

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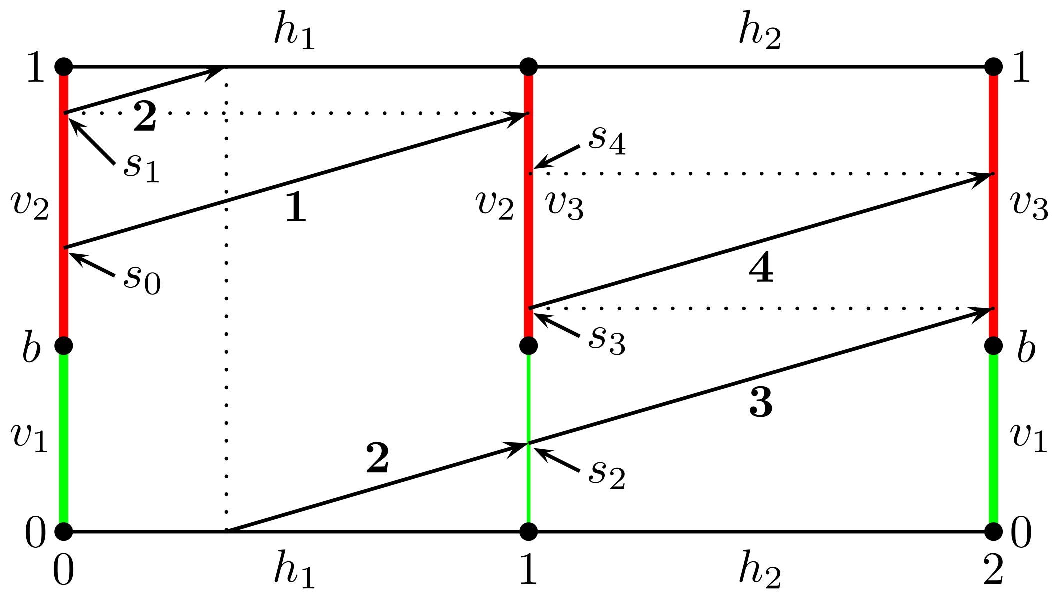

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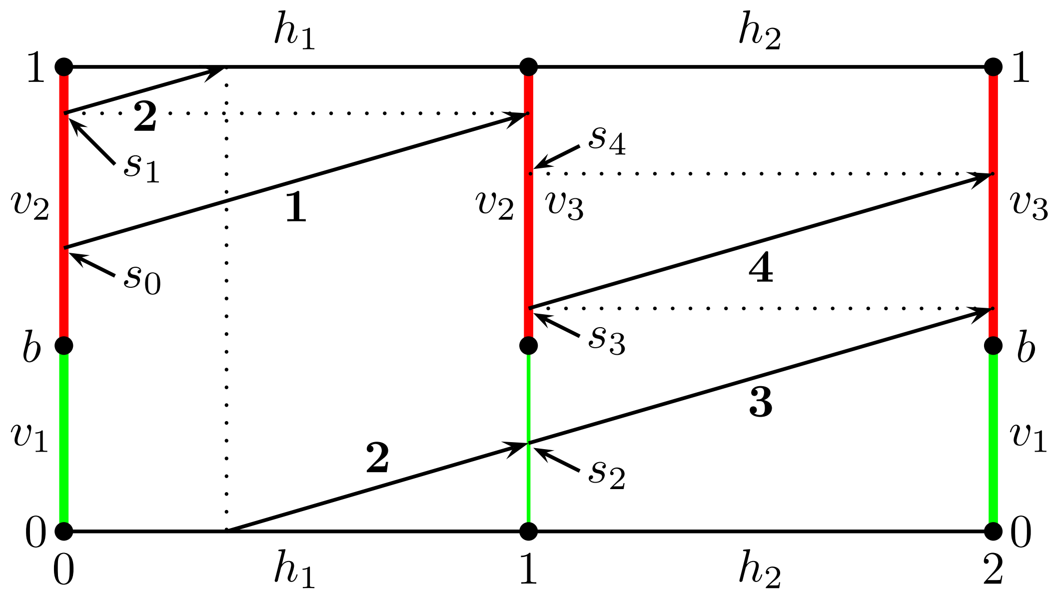

2-square-b surface

when is this α-geodesic equidistributed ?

special case $b = \{m\alpha\}$, where $m \in \mathbb{Z}$

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not difficult to show that we only need to consider $m > 0$

special case $b = \{m\alpha\}$: simple way to determine left or right

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$s_q = \{\tau + q\alpha\},\ q = 0, 1, 2, 3, \ldots$

parity of $\Psi(\alpha; \tau; b; N) = |\{q = 0, 1, \ldots, N-1 : 0 \leqslant s_q < b\}|$

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general : $\Psi(\alpha;\tau;b;N) = \begin{cases} \lceil \{\tau\} + (N-1)\alpha \rceil & \text{if } 0 \leqslant \{\tau\} < b \\ \lfloor \{\tau\} + (N-1)\alpha \rfloor & \text{if } b \leqslant \{\tau\} < 1 \end{cases}$

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special case $0 < b = \{2\alpha\} < 1$ and $\tau = 0$:

special case $b = \{m\alpha\}$: simple way to determine left or right

$s_q = \{\tau + q\alpha\}$, $q = 0, 1, 2, 3, \ldots$

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special case $0 < b = \{2\alpha\} < 1$ and $\tau = 0$:

apply $(*)$ to $s_0, s_2, s_4, s_6, \ldots$ and $s_1, s_3, s_5, s_7, \ldots$

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increasingly complicated

Veech (1969) :

α badly approximable

$b \neq \{m\alpha\}$ for any $m \in \mathbb{Z}$

$\mathcal{L}$ − half-infinite α-geodesic on 2-square-b surface

$\Rightarrow$ $\mathcal{L}$ evenly distributed between the two squares

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$\mathcal{L}$ − half-infinite α-geodesic on 2-square-b surface

$\Rightarrow$ $\mathcal{L}$ evenly distributed between the two squares

excludes a countable set of values of b

what happens when $b = \{m\alpha\}$ for some $m \in \mathbb{Z}$?

α irrational

$m = 1$: $0 < b = \alpha < 1$:

$m = 1$: $0 < b = \alpha < 1$: equidistributed for any α-geodesic

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$m = 2$: $0 < b = \{2\alpha\} < 1$:

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$m = 1$: $0 < b = \alpha < 1$: equidistributed for any α-geodesic

$m = 2$: $0 < b = \{2\alpha\} < 1$:

$0 < \alpha < \frac{1}{2}$:

α irrational

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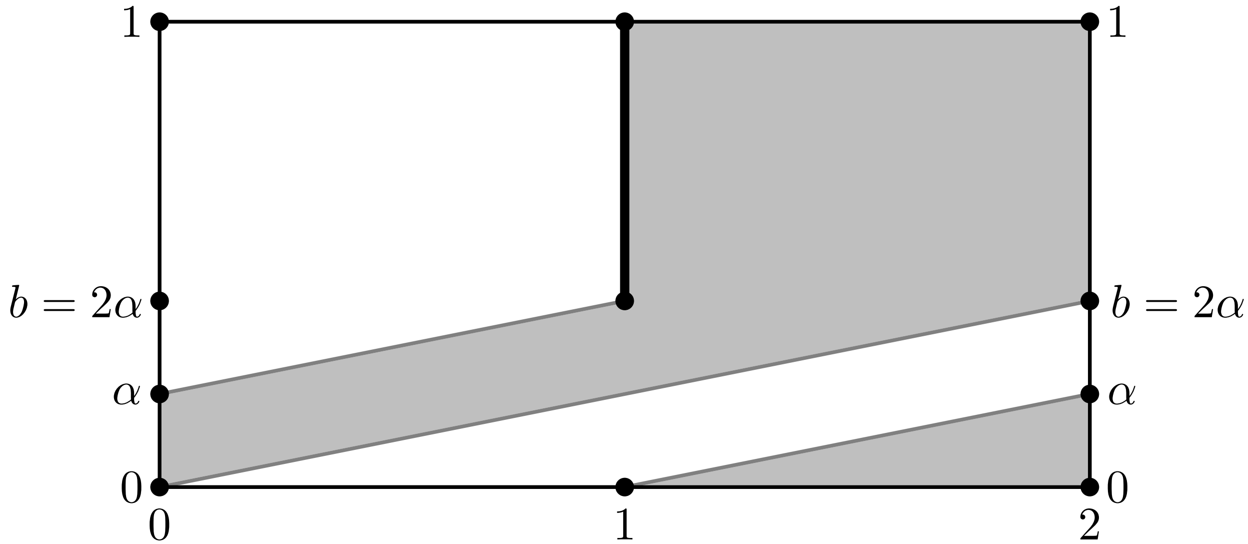

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$0 < \alpha < \frac{1}{2}$: not dense or equidistributed for any α-geodesic

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$\frac{1}{2} < \alpha < 1$: equidistributed for any α-geodesic

$m = 3$: $0 < b = \{3\alpha\} < 1$:

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$m = 3$: $0 < b = \{3\alpha\} < 1$: equidistributed for any α-geodesic

$m = 4$: $0 < b = \{4\alpha\} < 1$:

α irrational

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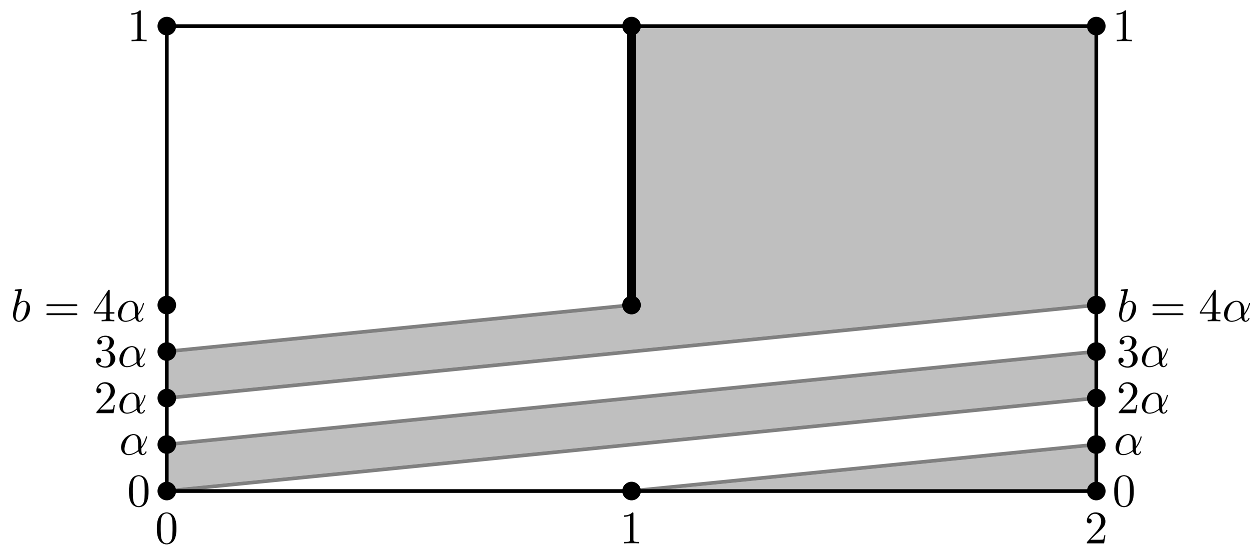
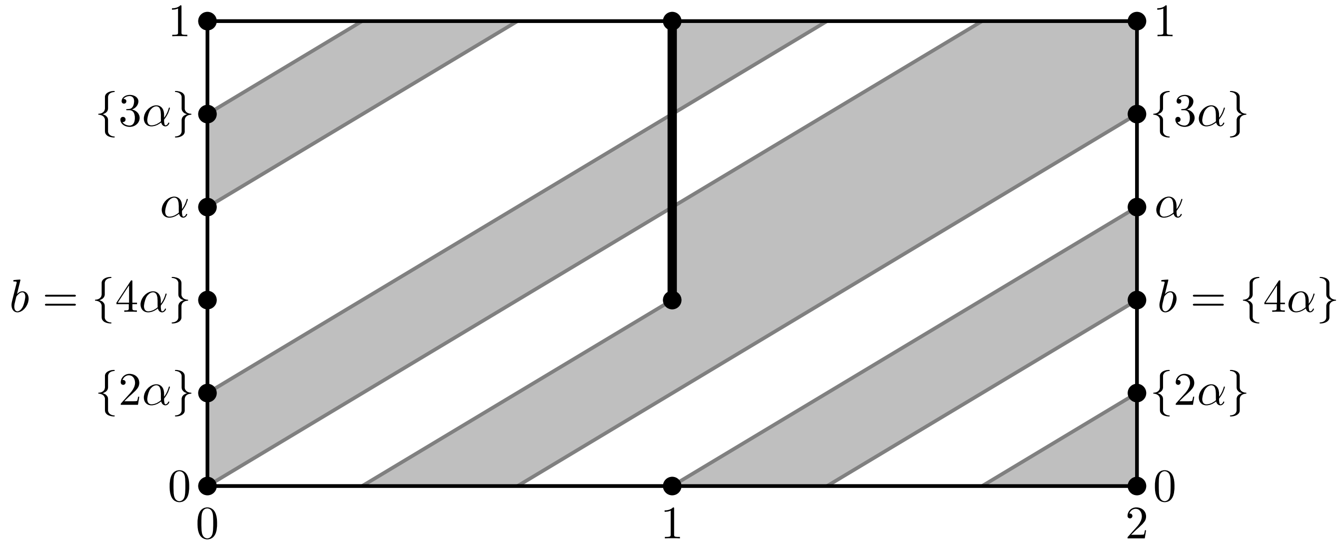
$0 < \alpha < \frac{1}{4}$:



α irrational

$m = 4$: $0 < b = \{4\alpha\} < 1$:

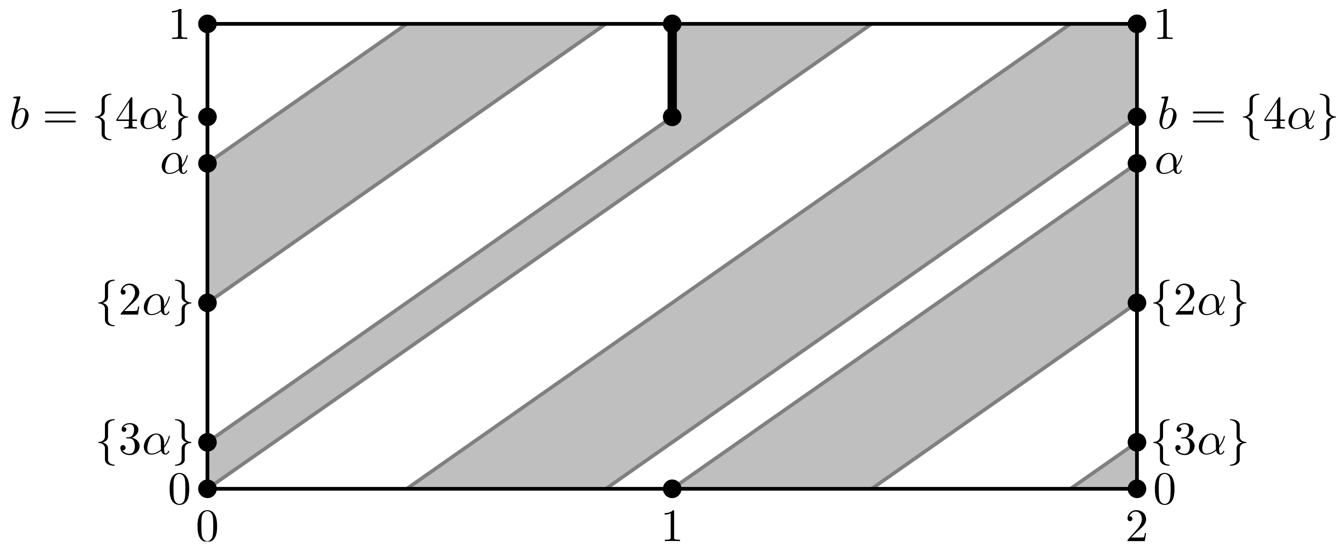
$\frac{1}{2} < \alpha < \frac{2}{3}$:



α irrational

$m = 4\,:\; 0 < b = \{4\alpha\} < 1\;:$

$\frac{2}{3} < \alpha < \frac{3}{4}\;:$



α irrational

$m = 4$: $0 < b = \{4\alpha\} < 1$:

$0 < \alpha < \frac{1}{4}$ or $\frac{1}{2} < \alpha < \frac{3}{4}$: not dense or equidistributed for any α-geodesic

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case study seems hopelessly complicated and mysterious

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there is a simple underlying rule

$PP(m;\alpha) = \#\{q = 1, \ldots, m : \{q\alpha\} < \alpha\}$

parity parameter

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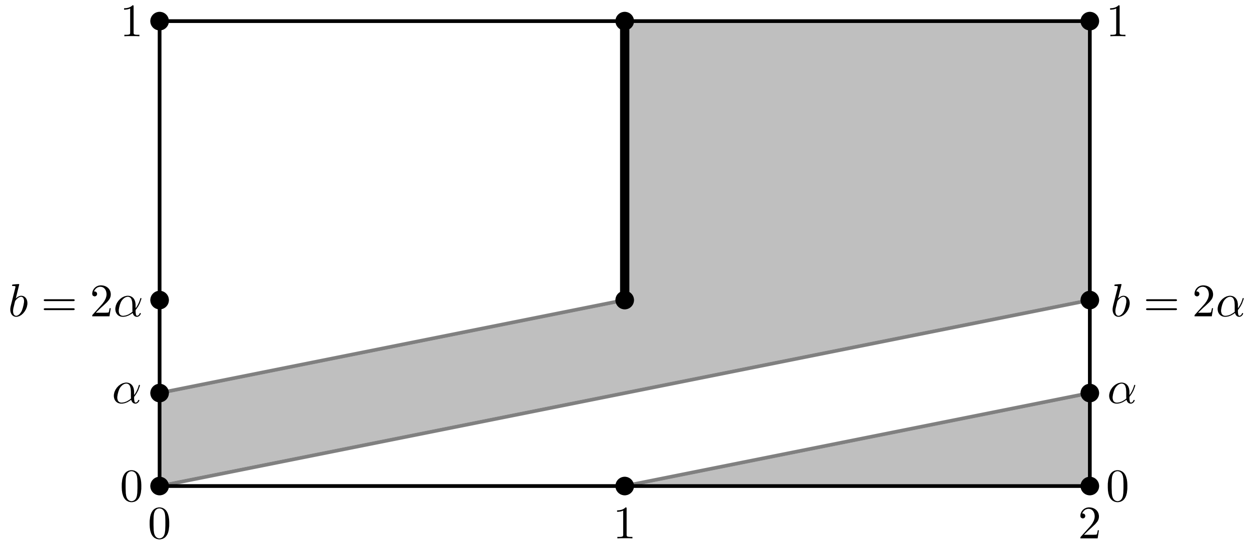
$m = 2$ and $0 < \alpha < \frac{1}{2}$:



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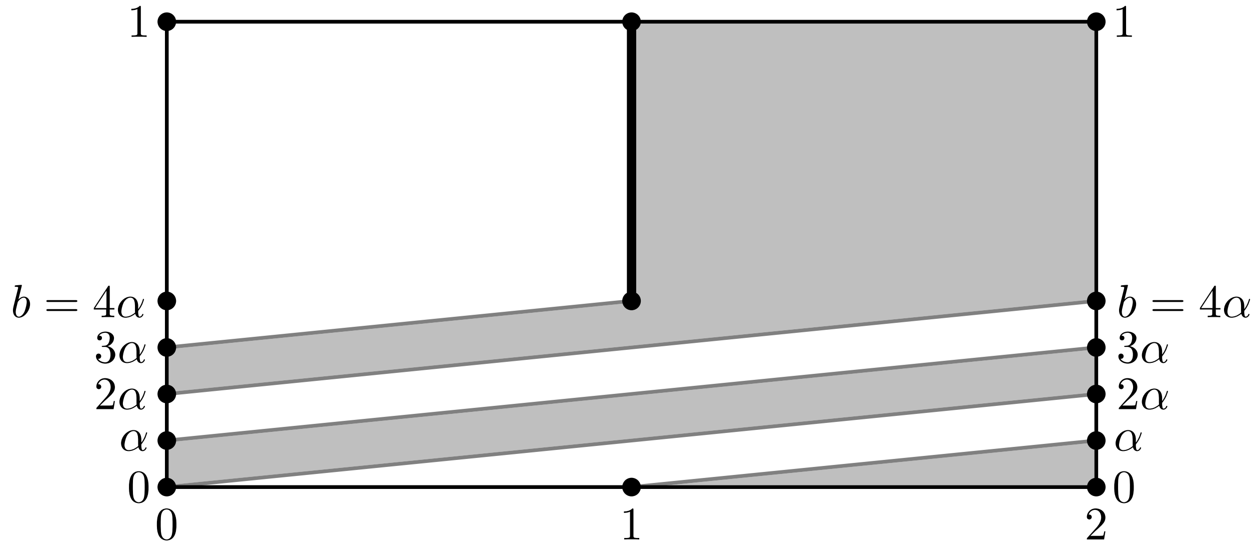

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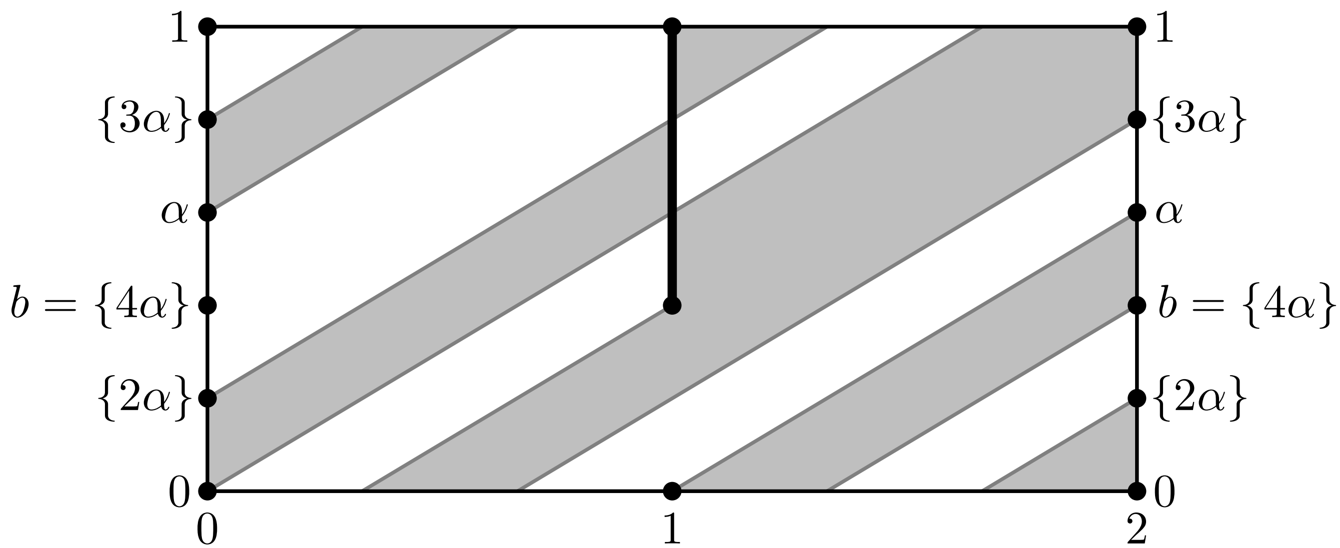
$m = 4$ and $\frac{1}{2} < \alpha < \frac{2}{3}$:



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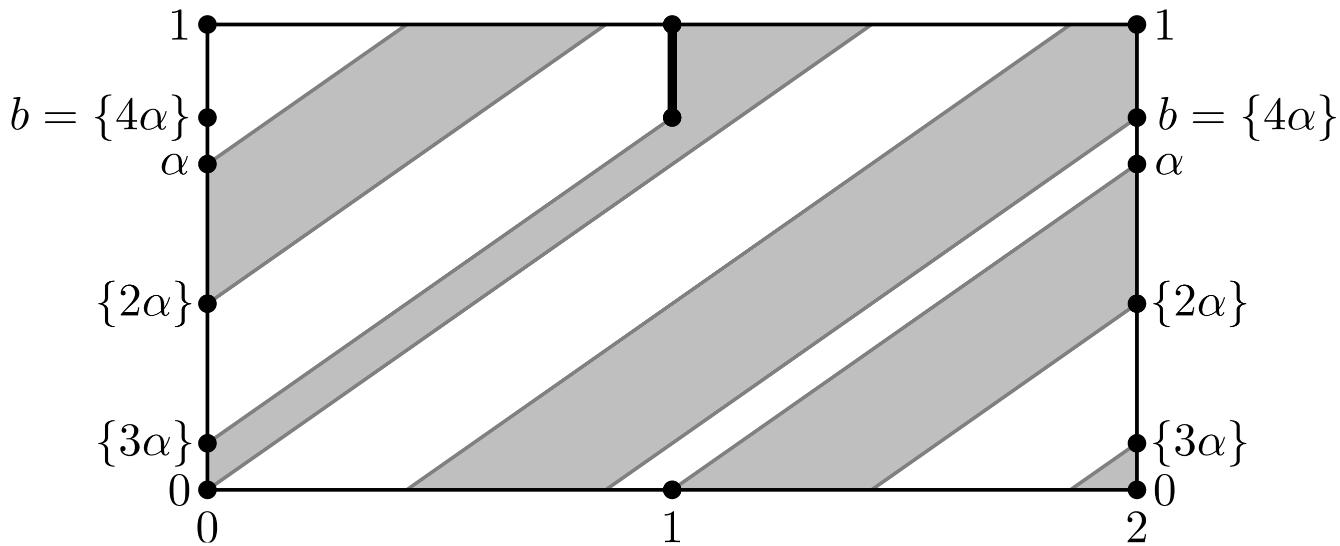
$m = 4$ and $\frac{2}{3} < \alpha < \frac{3}{4}$:



$PP(m;\alpha) = \#\{q = 1, \ldots, m : \{q\alpha\} < \alpha\}$

parity parameter

$m = 2 : \; PP(2;\alpha) = \; 0 \;\text{ if } 0 < \alpha < \frac{1}{2}$

$$m = 4 : \; PP(4;\alpha) = \begin{cases} 0 & \text{if } 0 < \alpha < \frac{1}{4} \\ 2 & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ 2 & \text{if } \frac{2}{3} < \alpha < \frac{3}{4} \end{cases}$$

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Double-Even Criterion : m and $PP(m;\alpha)$ are both even

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parity parameter

Double-Even Criterion : m and $PP(m;\alpha)$ are both even

BCY (2021) : $b = \{m\alpha\}$ for integer $m > 0$

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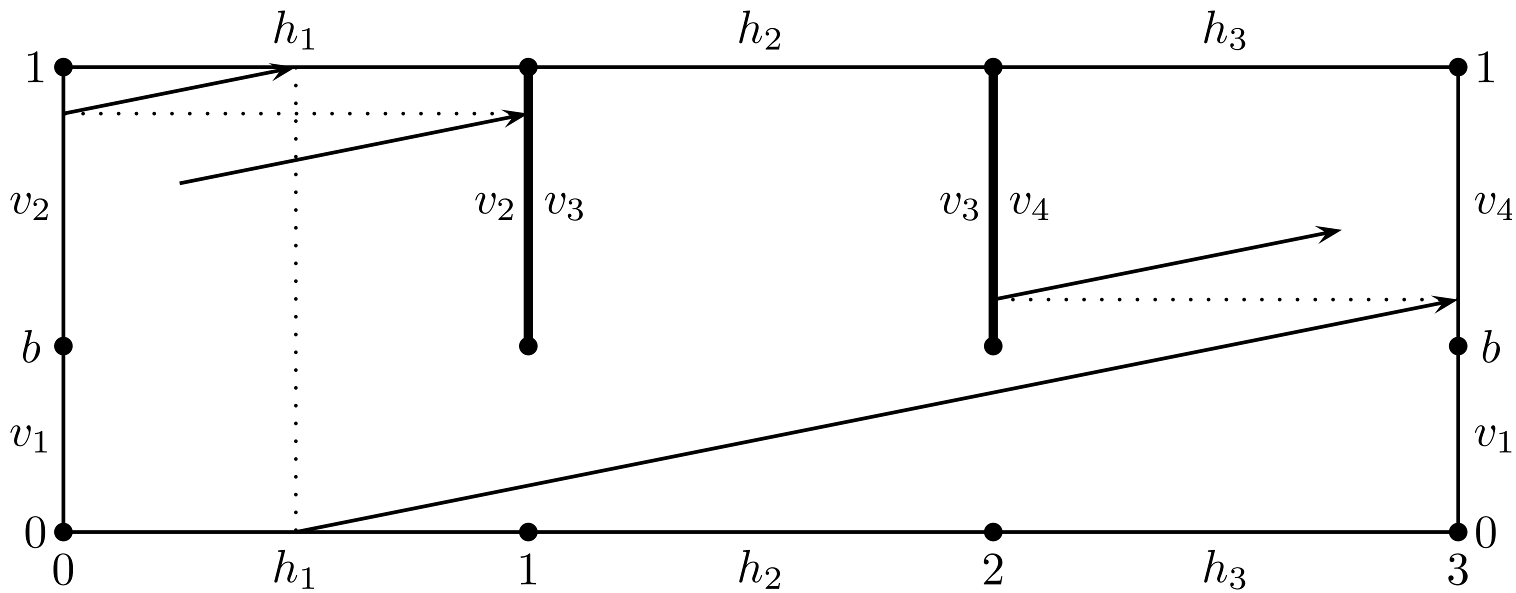
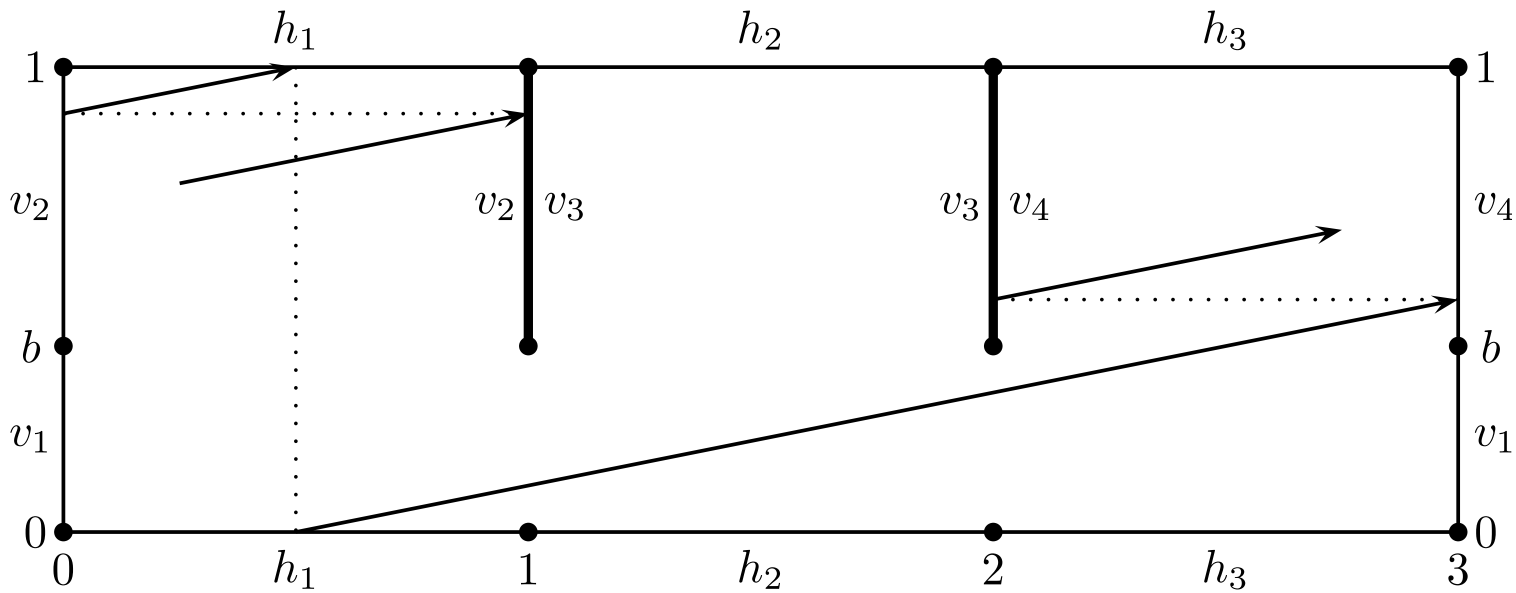
3-square-b surface



3-square-b surface



n-square-b surface for any integer $n \geqslant 2$

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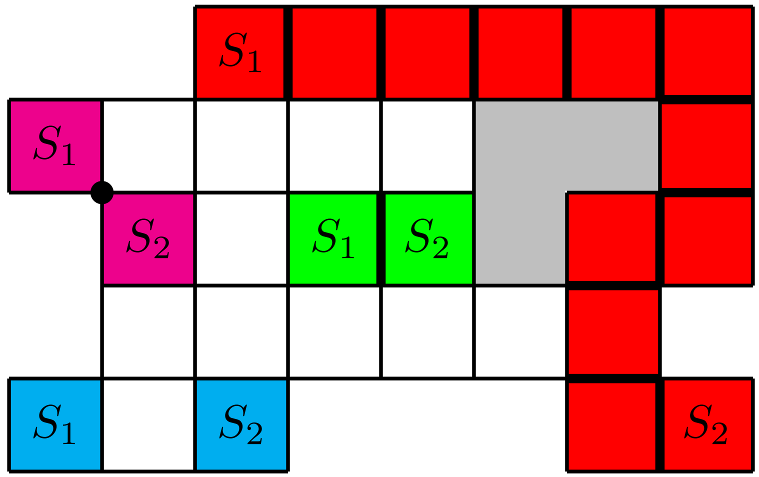
$\mathcal{L}$ − half-infinite α-geodesic on n-square-b surface

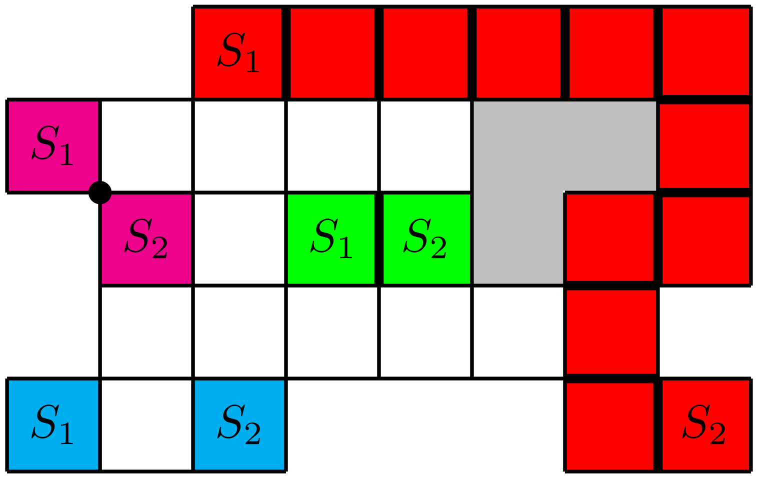
$\Rightarrow$ $\mathcal{L}$ equidistributed finite polysquare region – with finitely many unit size squares



disjoint or common vertex or common edge

chain with common edges

finite polysquare region – with finitely many unit size squares



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horizontal edge identification + vertical edge identification

$\hookrightarrow$ finite polysquare surface

Gutkin (1984) $\oplus$ Veech (1987) :

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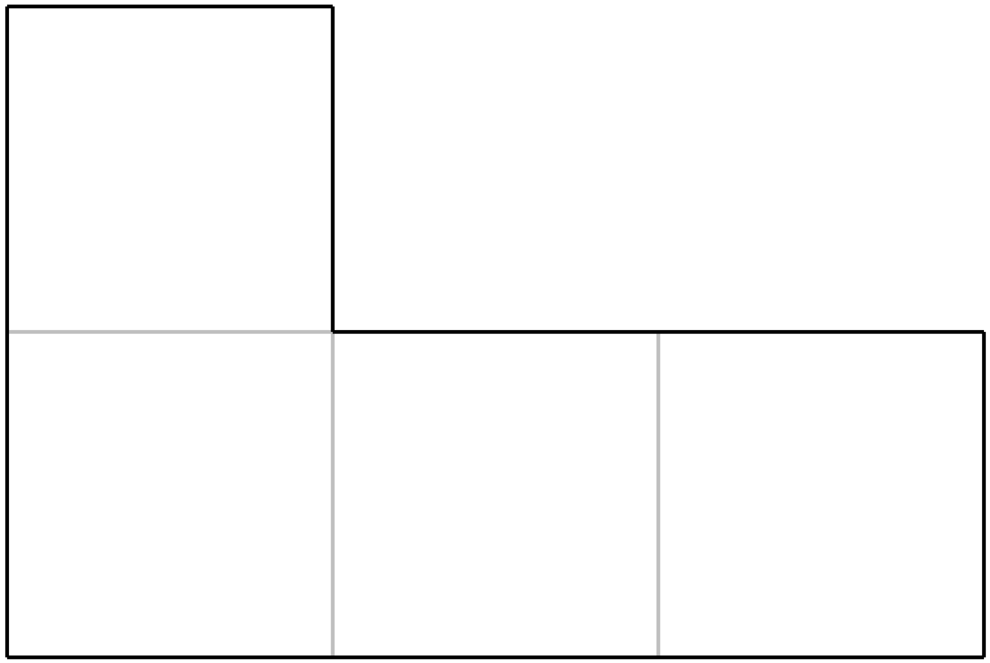
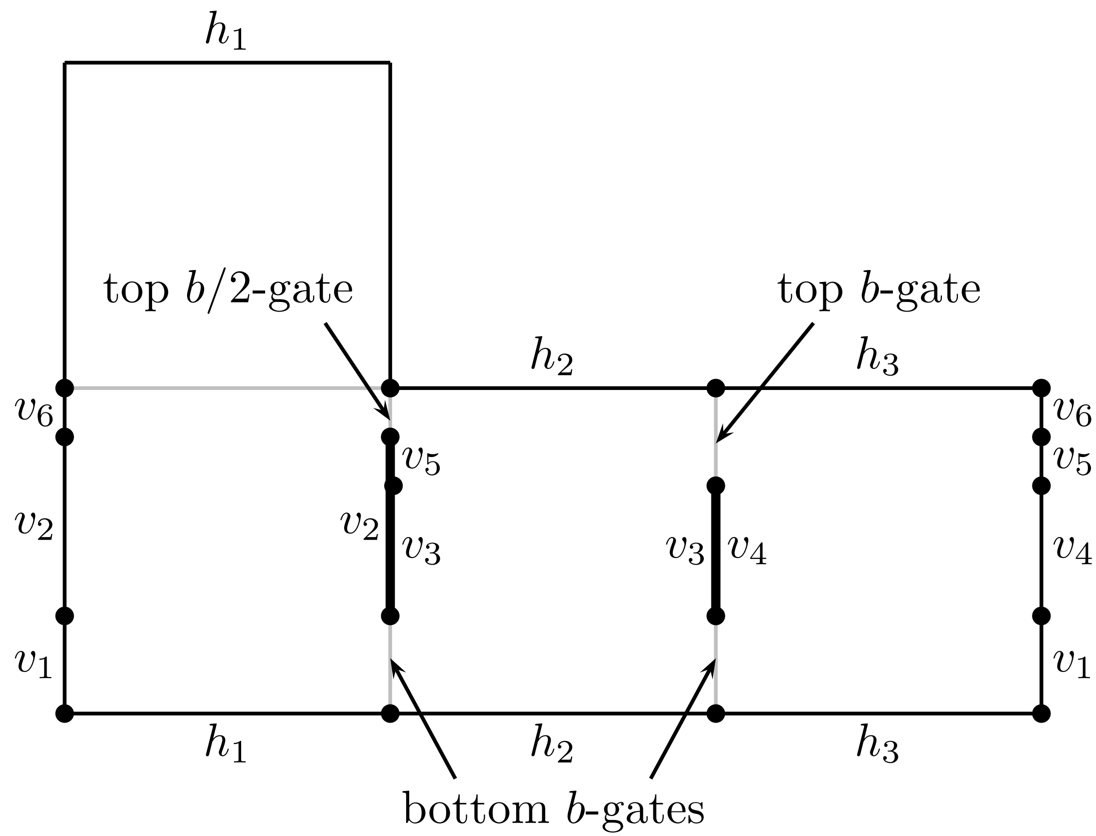
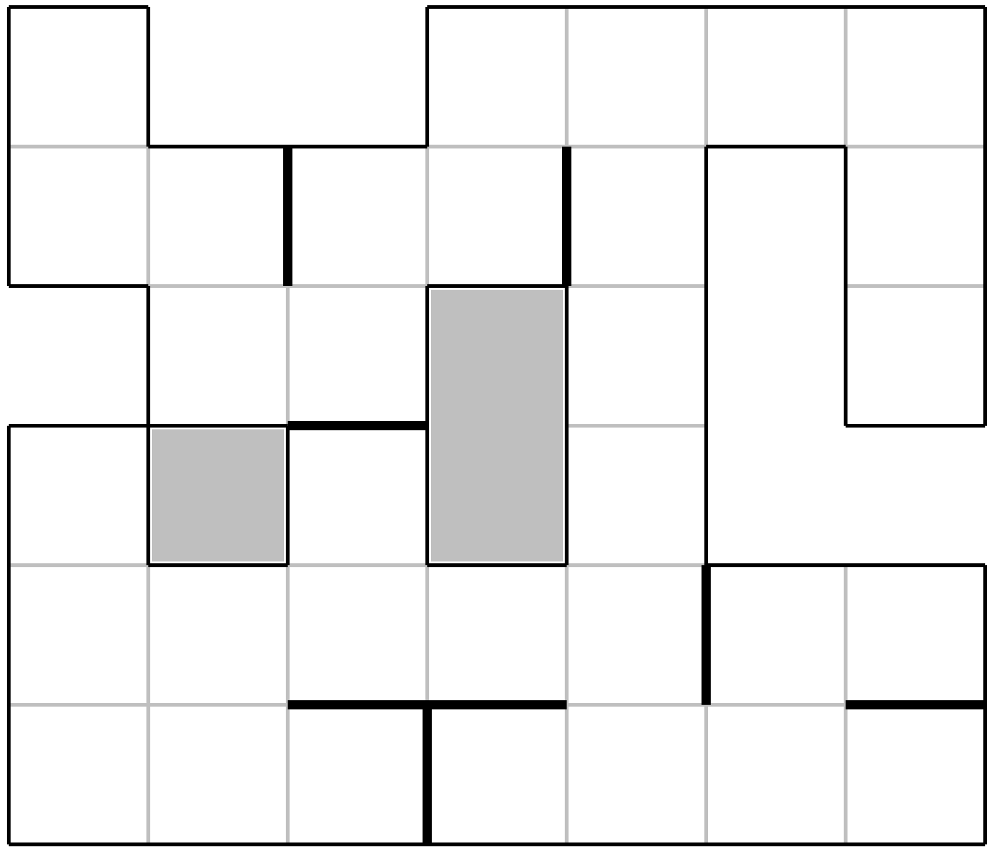
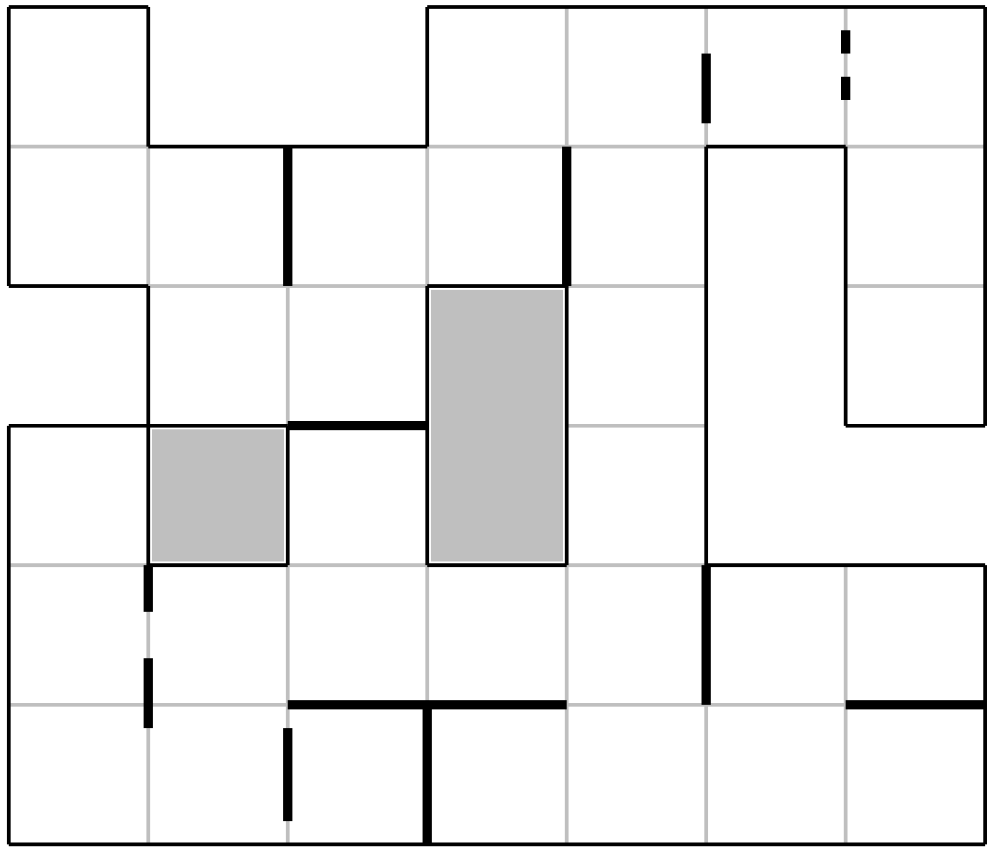
uniform-periodic dichotomy




h_1
top $b/2$-gate
top b-gate
h_2
h_3
v_6
v_5
v_2
v_3
v_4
v_1
h_1
h_2
h_3
bottom b-gates





finite polysquare b-rational translation surface $\mathcal{P}$

division numbers $\{r_i b\}$, $i = 1, \ldots, R$, where each $r_i \in \mathbb{Q}$

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barriers and gates between division points

modification of edge identifications to ensure we have a surface

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$\{r_i b\} \neq \{m\alpha\}$ for any $i = 1, \ldots, R$ and $m \in \mathbb{Z} \setminus \{0\}$

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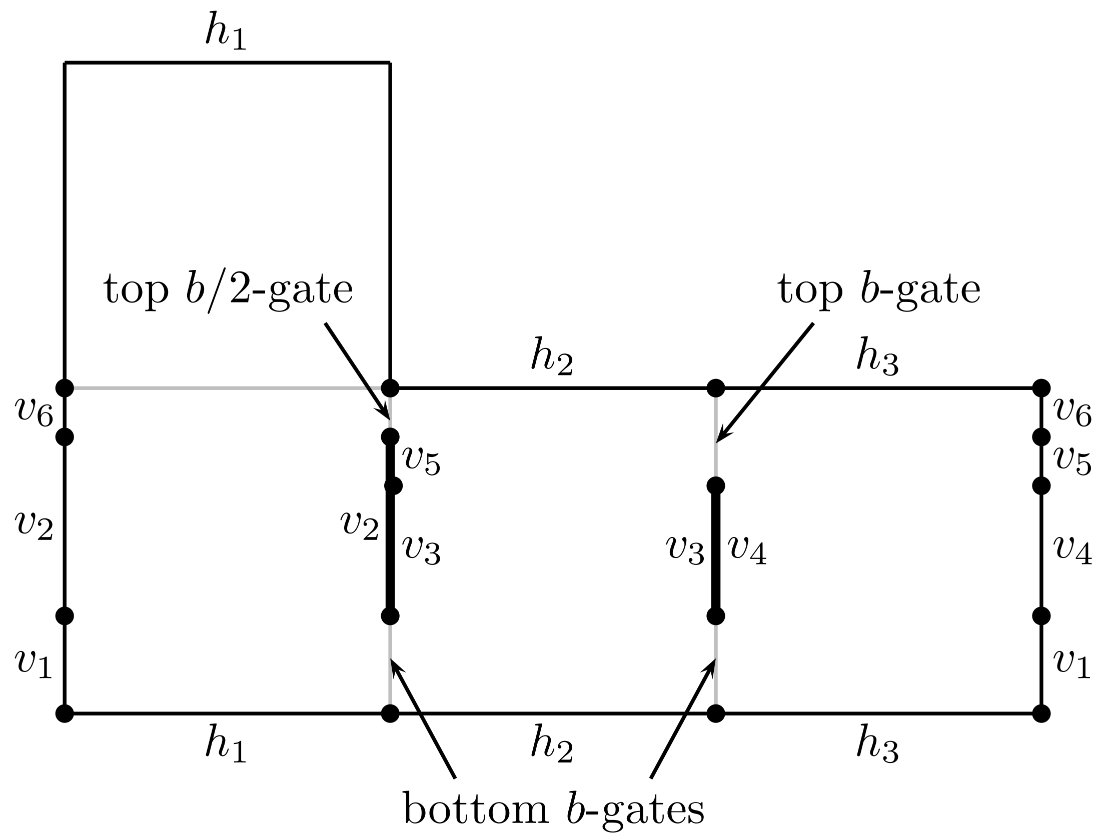
h_1
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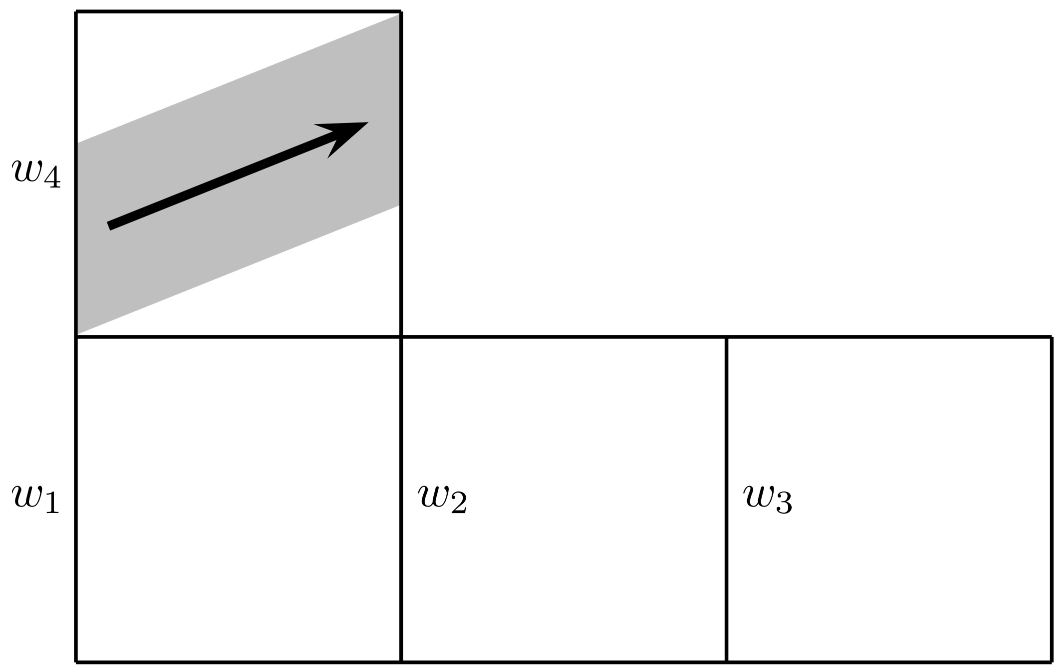
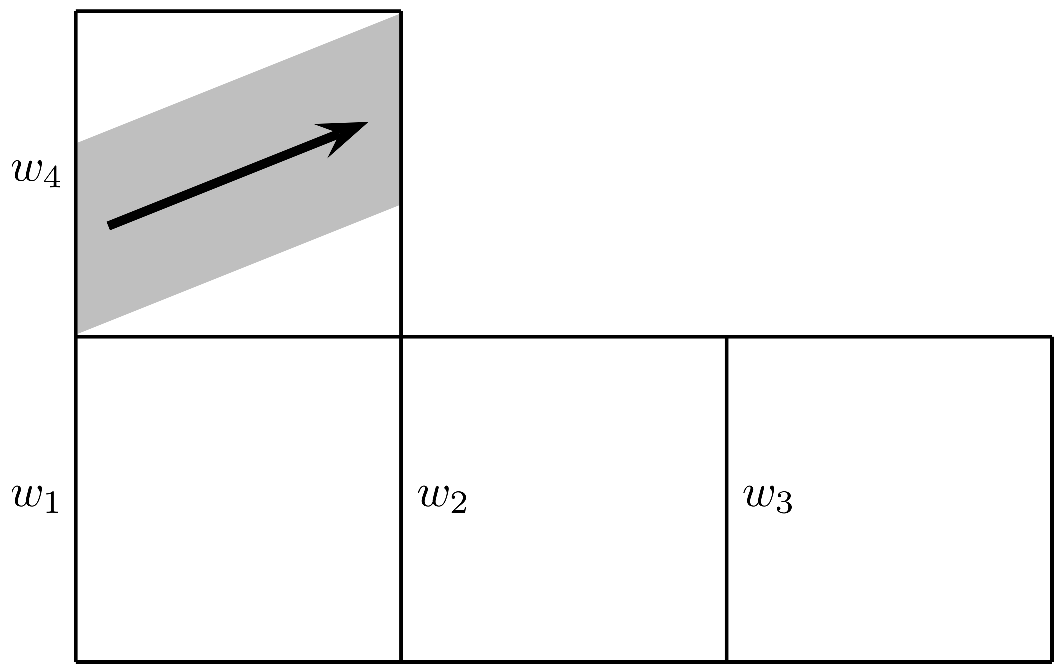
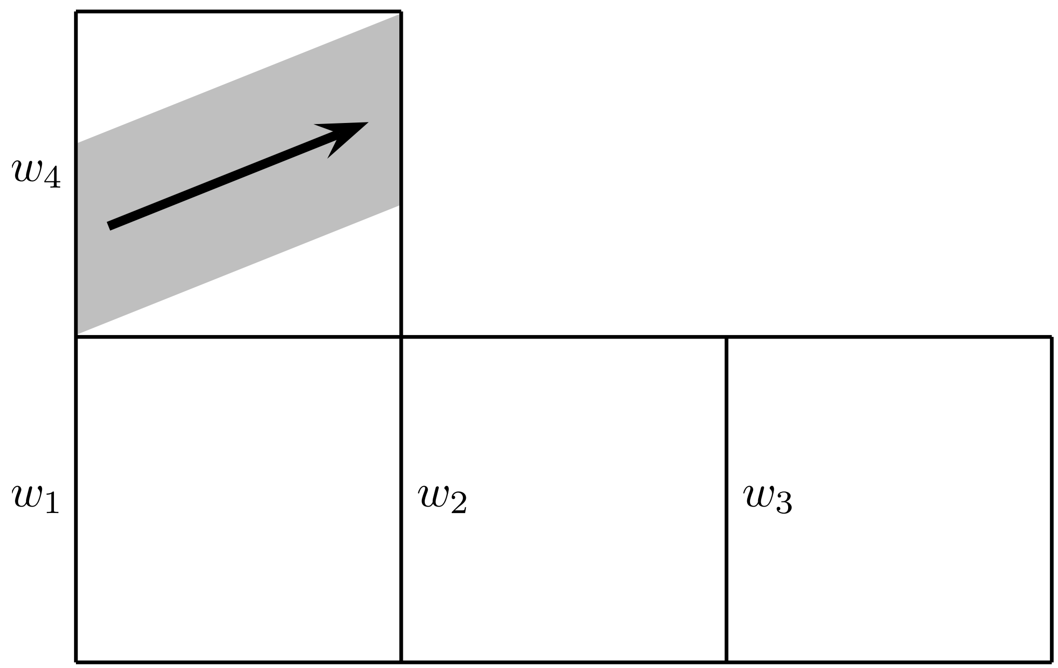

w_4
w_1
w_2
w_3



assume for simplicity that $0 < b < \alpha < \frac{1}{2}$



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$w_4[0, 1 - \alpha) \mapsto w_4[\alpha, 1)$

$$
\begin{aligned}
w_1[0, 1-\alpha-\tfrac{b}{2}) &\mapsto w_1[\alpha, 1-\tfrac{b}{2}) \\
w_1[1-\alpha-\tfrac{b}{2}, 1-\alpha) &\mapsto w_2[1-\tfrac{b}{2}, 1) \\
w_1[1-\alpha, 1) &\mapsto w_4[0, \alpha) \\
w_2[0, 1-\alpha-b) &\mapsto w_2[\alpha, 1-b) \\
w_2[1-\alpha-b, 1-\alpha) &\mapsto w_3[1-b, 1) \\
w_2[1-\alpha, 1-\alpha+b) &\mapsto w_3[0, b) \\
w_2[1-\alpha+b, 1) &\mapsto w_2[b, \alpha) \\
w_3[0, 1-\alpha-b) &\mapsto w_3[\alpha, 1-b) \\
w_3[1-\alpha-b, 1-\alpha-\tfrac{b}{2}) &\mapsto w_2[1-b, 1-\tfrac{b}{2}) \\
w_3[1-\alpha-\tfrac{b}{2}, 1-\alpha) &\mapsto w_1[1-\tfrac{b}{2}, 1) \\
w_3[1-\alpha, 1-\alpha+b) &\mapsto w_1[0, b) \\
w_3[1-\alpha+b, 1) &\mapsto w_3[b, \alpha) \\
w_4[0, 1-\alpha) &\mapsto w_4[\alpha, 1) \\
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w_4[1-\alpha+b, 1) &\mapsto w_1[b, \alpha)
\end{aligned}
$$

identity w_1, w_2, w_3, w_4 with $[0,1), [1,2), [2,3), [3,4)$ respectively

interval exchange transformation $T : [0,4) \to [0,4)$

$$T([0, 1-\alpha-\tfrac{b}{2})) = [\alpha, 1-\tfrac{b}{2})$$
$$T([1-\alpha-\tfrac{b}{2}, 1-\alpha)) = [2-\tfrac{b}{2}, 2)$$
$$T([1-\alpha, 1)) = [3, 3+\alpha)$$
$$T([1, 2-\alpha-b)) = [1+\alpha, 2-b)$$
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$$T([3-\alpha, 3-\alpha+b)) = [0, b)$$
$$T([3-\alpha+b, 3)) = [2+b, 2+\alpha)$$
$$T([3, 4-\alpha)) = [3+\alpha, 4)$$
$$T([4-\alpha, 4-\alpha+b)) = [1, 1+b)$$
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s square faces in general : $T : [0,s) \to [0,s)$

division numbers $\{r_i b\}$, $i = 1, \ldots, R$

singularities modulo one are 0, $1-\alpha$ and $\{r_i b - \alpha\}$, $i = 1, \ldots, R$

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Step 1 : Double-Even Criterion fails

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$\text{meas}(S_0) = 1$

irrational $\alpha \in (0,1)$

continued fraction $\alpha = [a_1, a_2, a_3, \ldots] = \dfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$

convergents $\dfrac{p_k}{q_k} = \dfrac{p_k(\alpha)}{q_k(\alpha)} = [a_1, \ldots, a_k],\ k = 1, 2, 3, \ldots$

$p_k \in \mathbb{Z}$ and $q_k \in \mathbb{N}$ are coprime

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < \alpha < \ldots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

$p_0 = 0,\ q_0 = 1,\ q_{-1} = 0$

$0, \alpha, 2\alpha, 3\alpha, \ldots, n\alpha$ modulo 1 $\hookrightarrow$ $(n+1)$-partition of unit torus $[0, 1)$

$\Rightarrow$ at most 3 different distances between neighboring partition points

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$n = \mu q_k + q_{k-1} + r$, with $1 \leqslant \mu \leqslant a_{k+1}$ and $0 \leqslant r < q_k$

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$n = q_{k+1} - 1 \Rightarrow \mu = a_{k+1} - 1$ and $r = q_k - 1$

$\Rightarrow$ 2-distance theorem : $\{d^*, d^{**}\} = \{\|q_k\alpha\|, \|q_{k+1}\alpha\| + \|q_k\alpha\|\}$

$\mathcal{A}_k(\alpha)$ – partition of $[0, 1)$ with q_{k+1} points

$\{q\alpha\}$, $-1 \leqslant q \leqslant q_{k+1} - 2$

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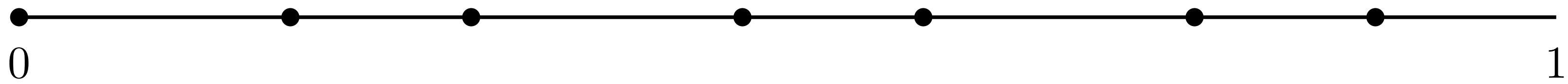
$m + 1$ long special intervals :



division points $\{q\alpha\}$, $q = -1, 0, 1, \ldots, m-1$

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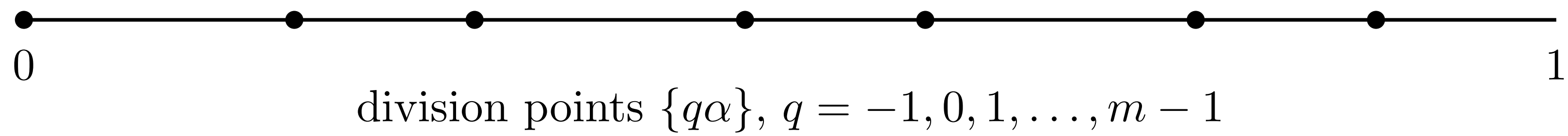
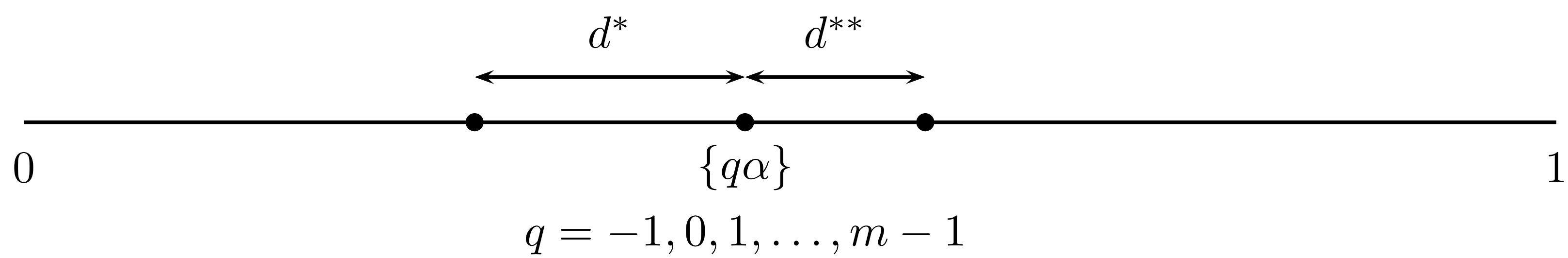


neighborhoods : $B(q)=(\{q\alpha\}-d^*,\{q\alpha\}+d^{**})$, $q=-1,0,1,\ldots,m-1$



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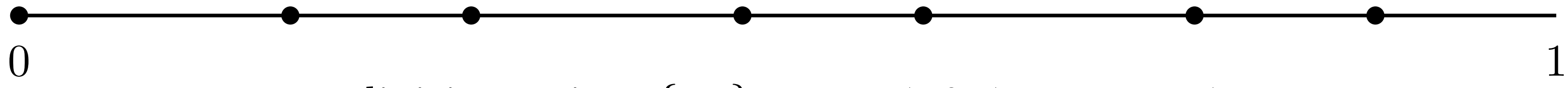

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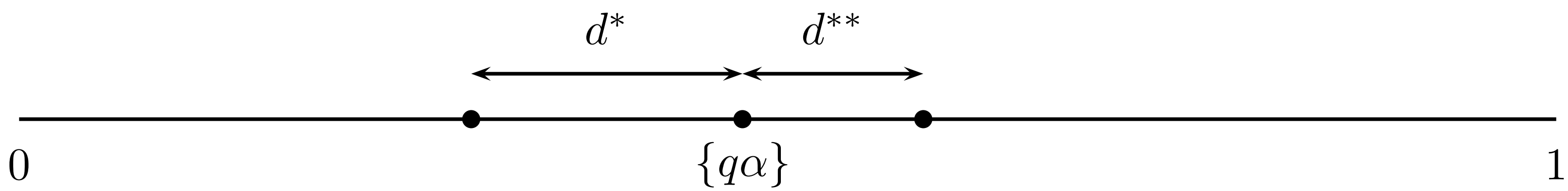

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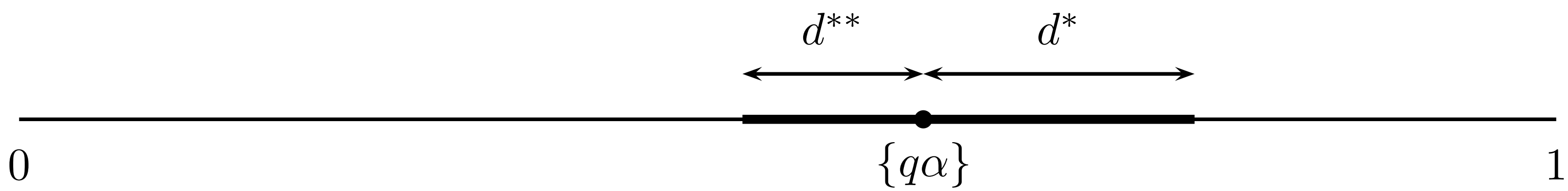
short intervals : $J_k(q)=(\{q\alpha\}-d^{**},\{q\alpha\}+d^{*})$, $q=m,\dots,q_{k+1}-2$



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properties :

∘ short intervals completely cover the $m + 1$ long special intervals

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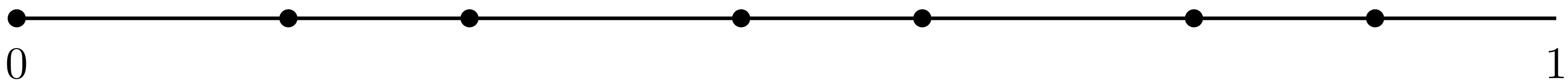

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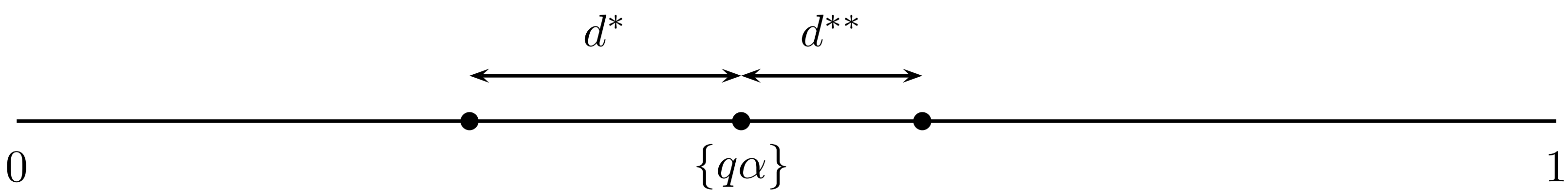

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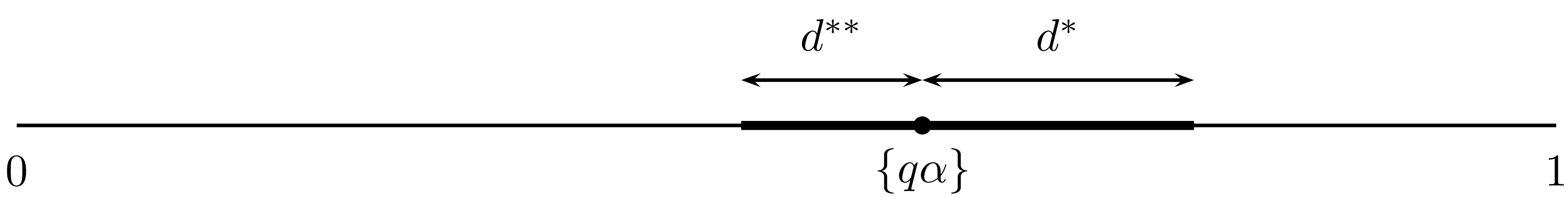

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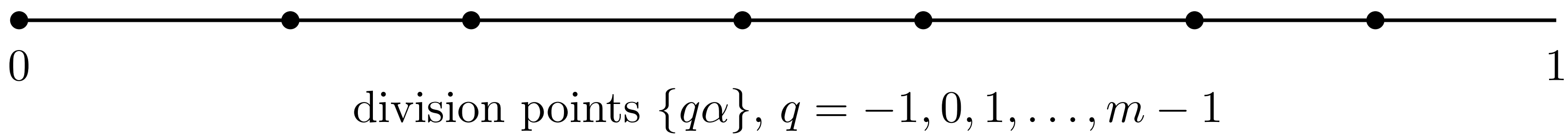

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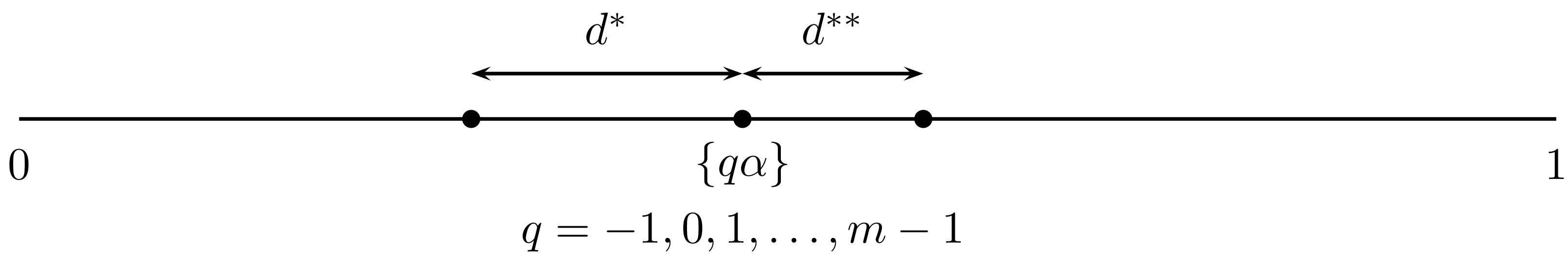

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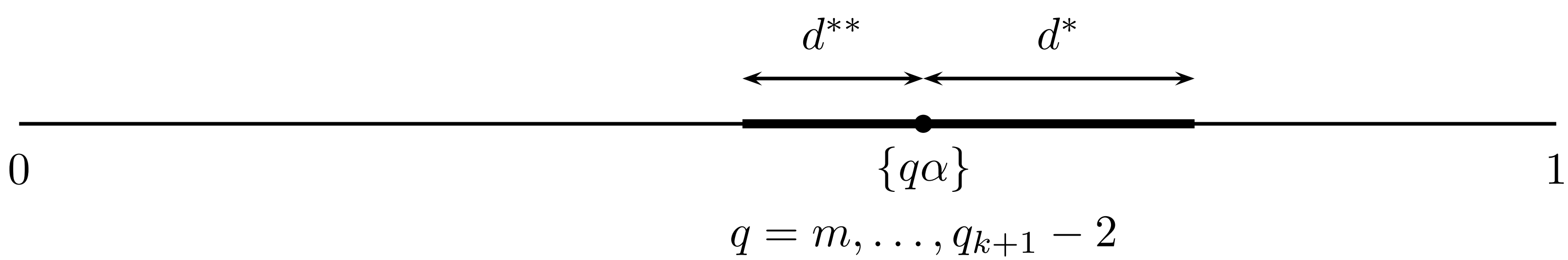

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from neighboring partition points exhibit substantial overlapping

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length$(J_k(q')\cap J_k(q''))\geqslant \min\{d^*,d^{**}\}=\|q_k\alpha\|$

length$(J_k(q))=d^*+d^{**}=2\|q_k\alpha\|+\|q_{k+1}\alpha\|<3\|q_k\alpha\|$

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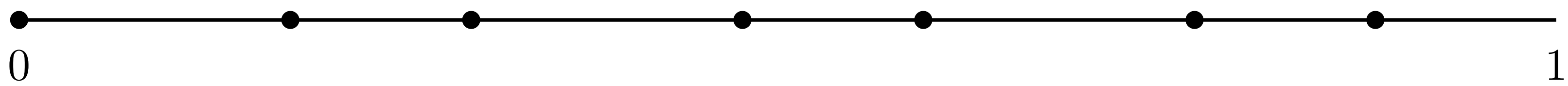

division points $\{q\alpha\}$, $q = -1, 0, 1, \ldots, m-1$

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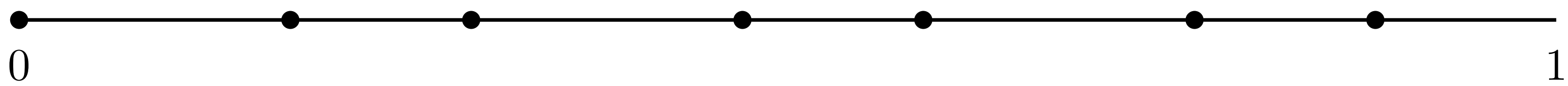

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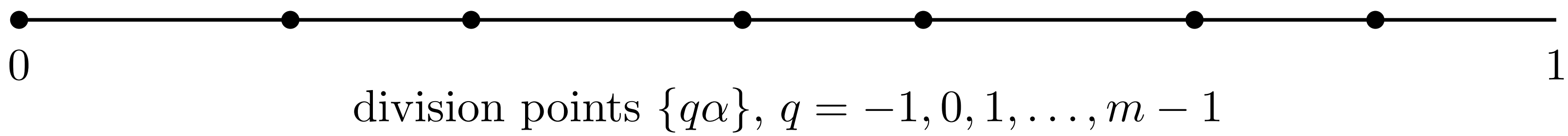

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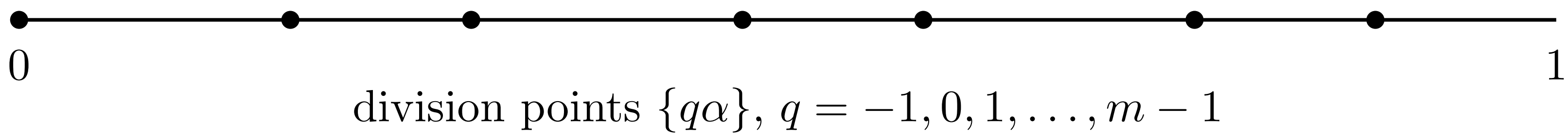

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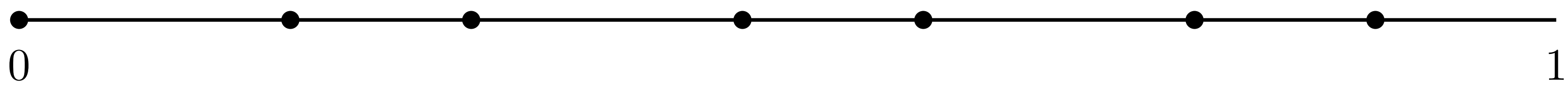

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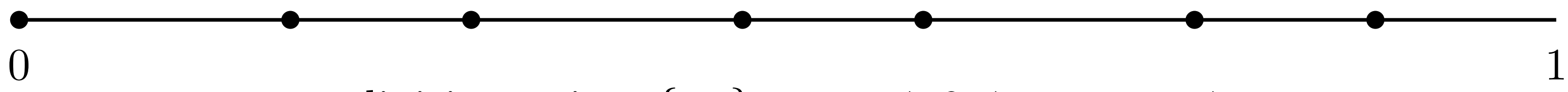

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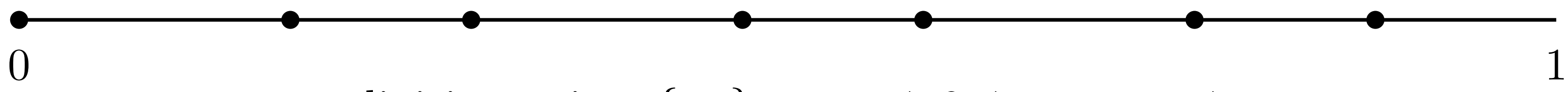

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all predominantly in S_0 or all predominantly outside S_0, so is $\mathcal{I}$

$\mathcal{I}$ predominantly in S_0 – color $\mathcal{I}$ blue

$\mathcal{I}$ predominantly outside S_0 – color $\mathcal{I}$ red

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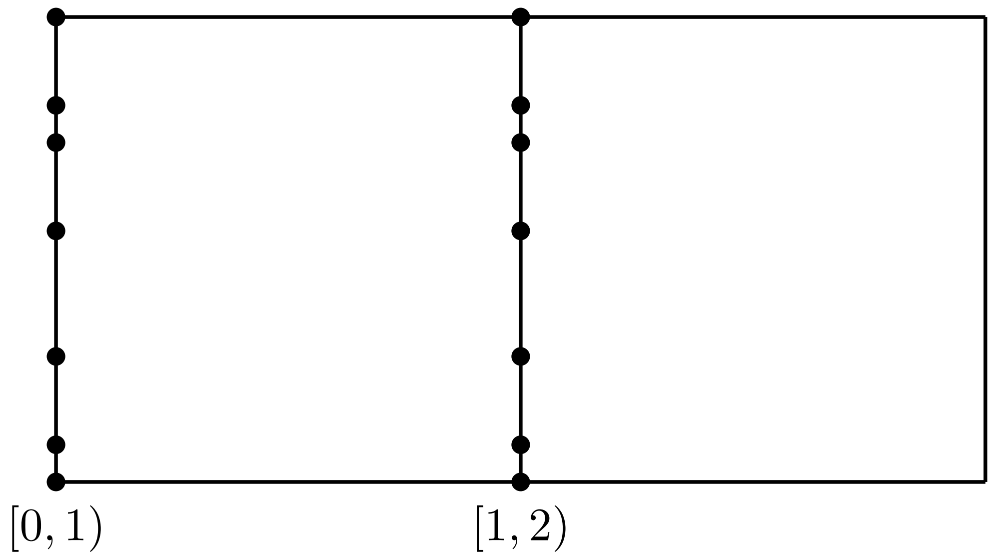

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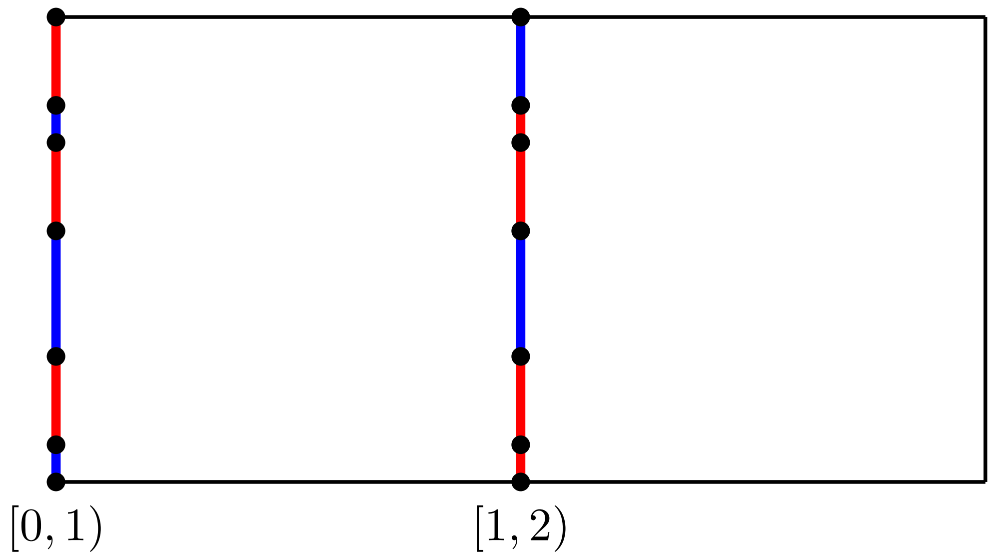

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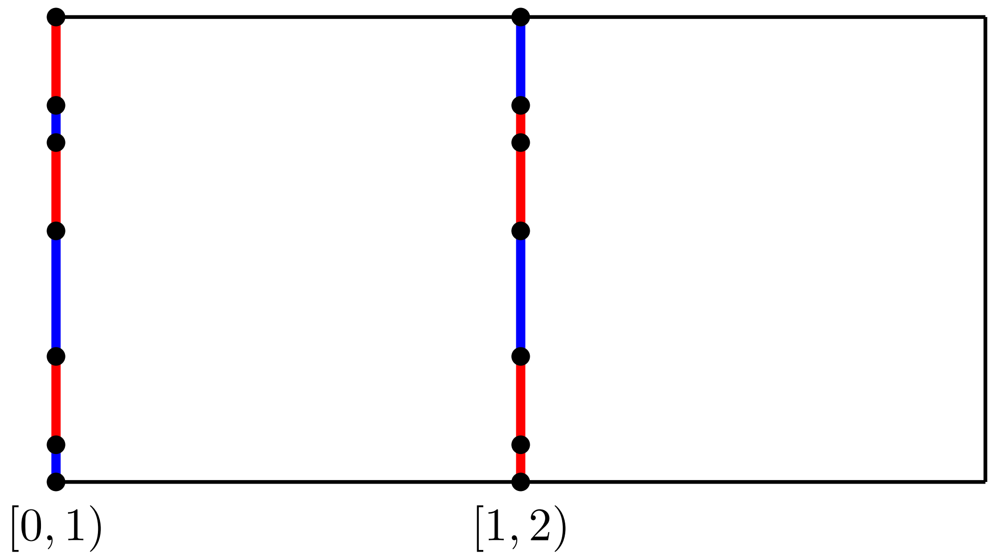
α-flow spreads this to a 2-coloring of the 2-square-b surface



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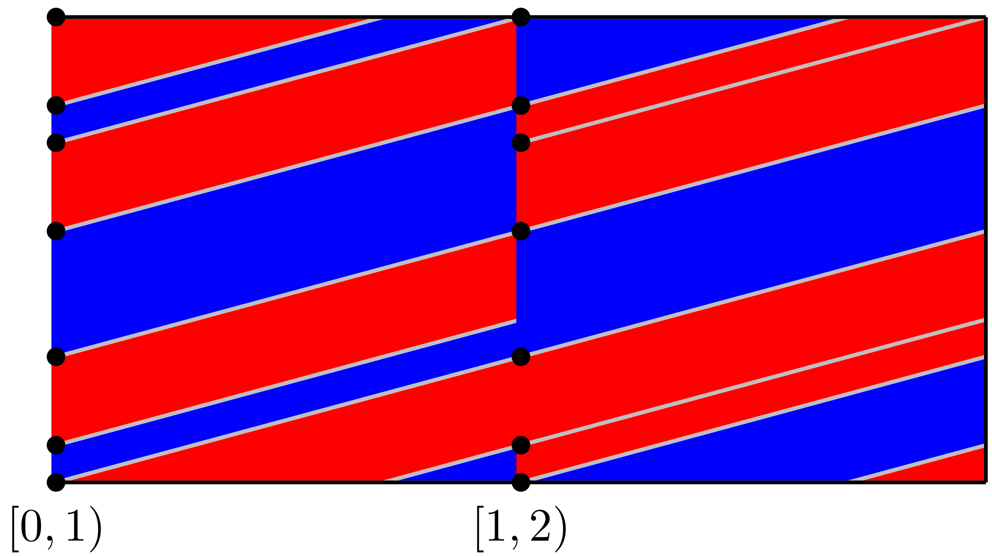

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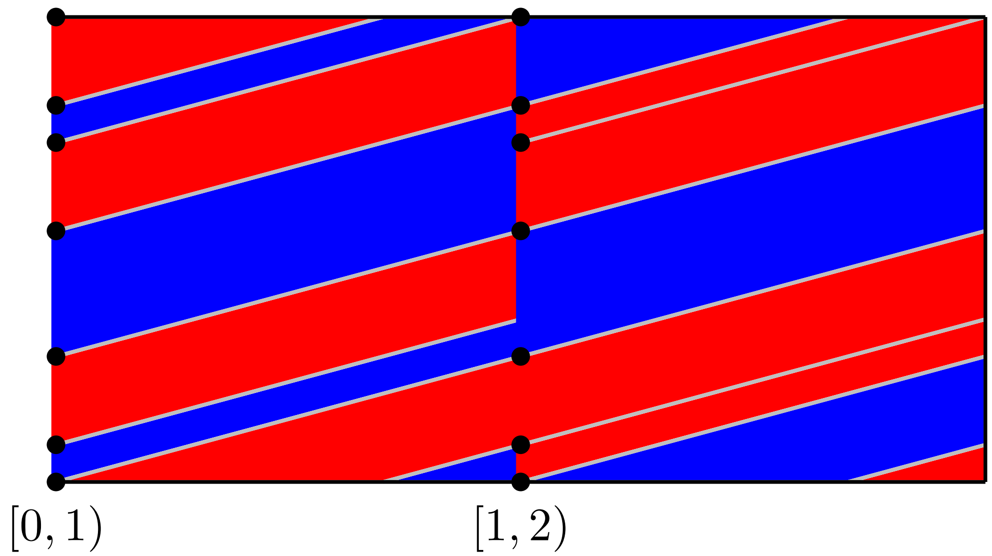

such a 2-coloring cannot exist if Double-Even Criterion fails

Step 1 : Double-Even Criterion fails

$\Rightarrow$ interval exchange transformation $T : [0, 2) \to [0, 2)$ is ergodic

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Step 2 : Extending ergodicity to unique ergodicity

Furstenberg + Birkhoff's ergodic theorem

Veech (1969) :

α badly approximable

$b \neq \{m\alpha\}$ for any $m \in \mathbb{Z}$

$\mathcal{L}$ − half-infinite α-geodesic on 2-square-b surface

$\Rightarrow$ $\mathcal{L}$ evenly distributed between the two squares

Veech (1969) :

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2-square-b surface with b irrational

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on the 2-square-b surface, the visit-densities of the squares do not exist

Masur–Smillie (2006) :

2-square-b surface with b irrational

$\Rightarrow$ uncountably many slopes α such that a half-infinite α-geodesic

with almost any starting point is not equidistributed

BCY (2021) :

$\varepsilon > 0$ arbitrarily small but fixed

irrational $\alpha \in (0,1)$

continued fraction $\alpha = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$

$$\sum_{i=1}^{\infty} \frac{1}{a_i} < \frac{\varepsilon}{300}$$

integer $C < \dfrac{300}{\varepsilon} - 1$

explicitly given gate-size $\beta_0 = \beta_0(\alpha)$

2-square-β_0 surface

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$\Rightarrow$ α-geodesic $\mathcal{L}_0(t)$, with explicitly given starting point

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2-square-β_0 surface

$\Rightarrow$ α-geodesic $\mathcal{L}_0(t)$, with explicitly given starting point

(i) sequence T_n^*, $n = 1, 2, 3, \ldots$, with $T_{n+1}^* > 2T_n^*$ such that

for every $b = 0, 1, \ldots, C$ apart from $b = 1$,

$$\frac{1}{T_n^*}|\{t \in [bT_n^*, (b+1)T_n^*] : \mathcal{L}_0(t) \in \mathsf{LS}(\beta_0)\}| > 1 - \varepsilon$$ left bias

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2-square-β_0 surface

$\Rightarrow$ α-geodesic $\mathcal{L}_0(t)$, with explicitly given starting point

(ii) sequence T_n^{**}, $n = 1, 2, 3, \ldots$, with $T_{n+1}^{**} > 2T_n^{**}$ such that

for every $b = 0, 1, \ldots, C$ apart from $b = 2$,

$$\frac{1}{T_n^{**}}|\{t \in [bT_n^{**}, (b+1)T_n^{**}] : \mathcal{L}_0(t) \in \mathsf{LS}(\beta_0)\}| > 1 - \varepsilon$$ left bias

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$\frac{1}{T_n^{**}} |\{t \in [2T_n^{**}, 3T_n^{**}] : \mathcal{L}_0(t) \in \mathrm{RS}(\beta_0)\}| > 1 - \varepsilon$ right bias

n given positive integer

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2-square-β_1 surface

$\Rightarrow$ α-geodesic $\mathcal{L}_1(t)$, with explicitly given starting point

(iii) sequence W_i, $i = 1, \ldots, n$, with $W_{i+1} > 2W_i$ such that

$$\frac{1}{W_i}|\{t \in [0, W_i] : \mathcal{L}_1(t) \in \mathsf{LS}(\beta_1)\}| > 1 - \varepsilon$$ left bias

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(iv) threshold W^* such that for all $W > W^*$,

$$\frac{1}{W}|\{t \in [0, W] : \mathcal{L}_1(t) \in \mathrm{LS}(\beta_1)\}| > \frac{2}{3} - \varepsilon$$ left bias

$$\Phi(\alpha;\beta;N) = |\{q = 0,\ldots,N-1 : \{q\alpha\} \in [0,\beta)\}|$$

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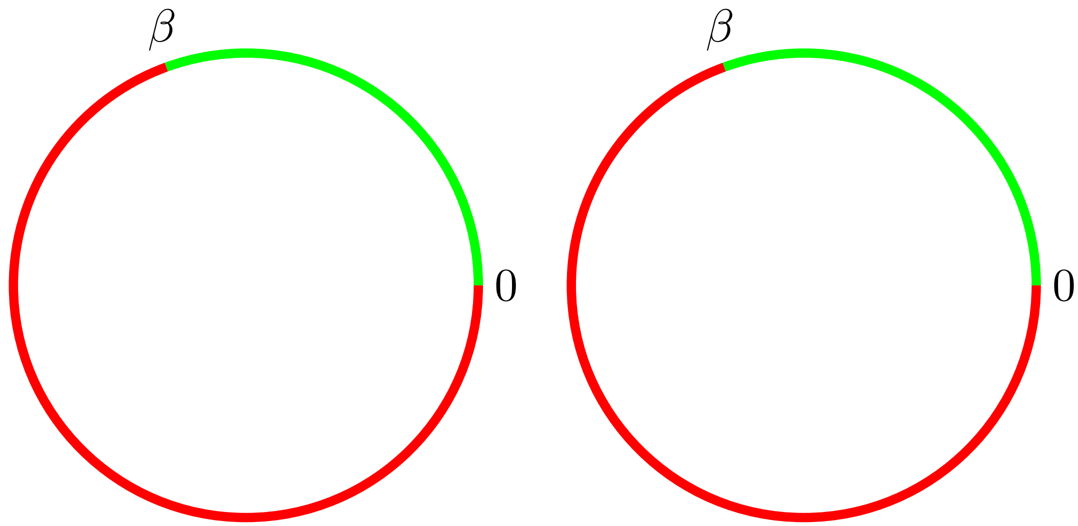

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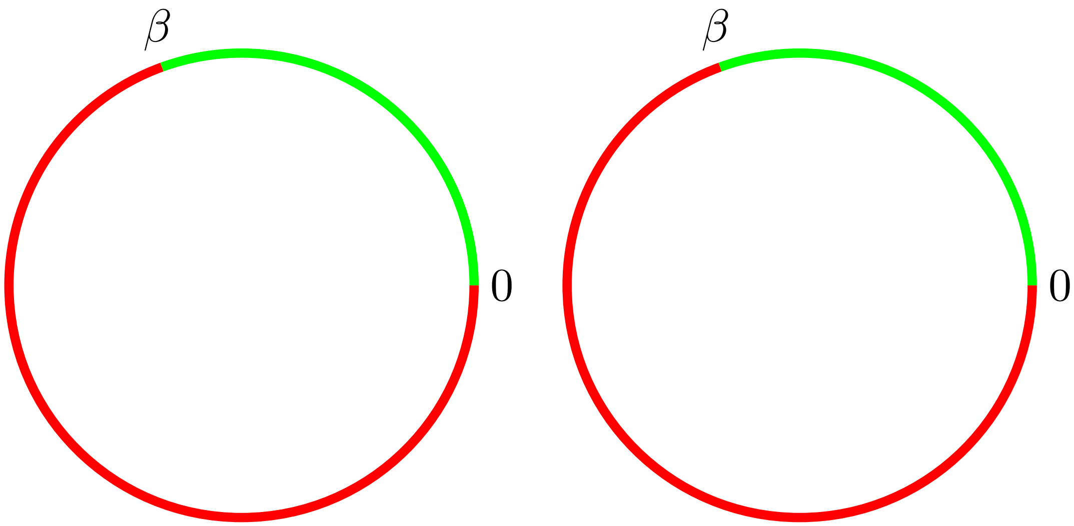

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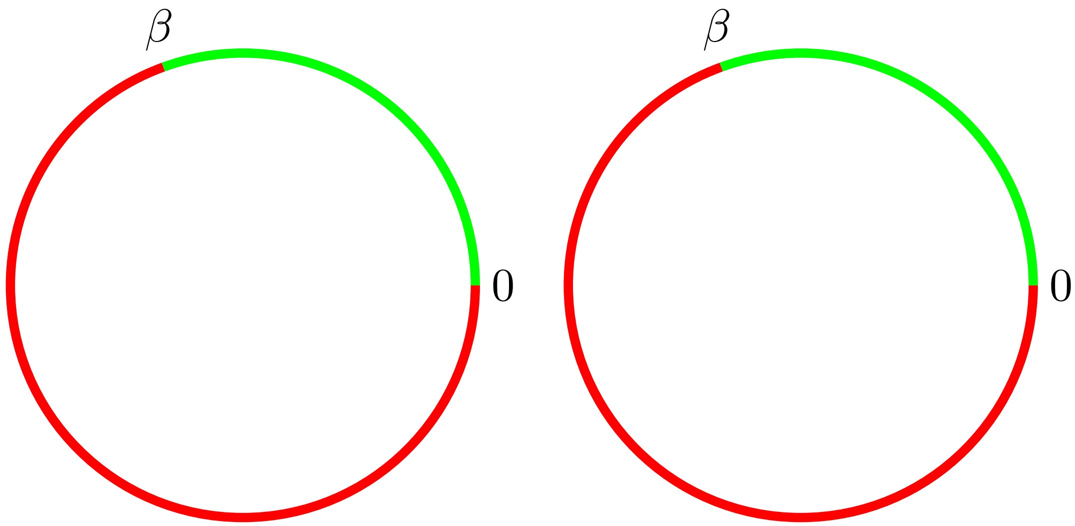

the parity of this function gives which circle

- $0 \leqslant q \leqslant N-1$

- $\{q\alpha\} \in [0,\beta)$

α-representation or Ostrowski representation

$q_0, q_1, q_2, \ldots$ sequence of denominators of convergents of α

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$q = \sum_{i=0}^{k} x_i q_i$ and $N = \sum_{i=0}^{k} b_i q_i$

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$q = \sum_{i=0}^{k} x_i q_i$ and $N = \sum_{i=0}^{k} b_i q_i$

$0 \leqslant q \leqslant N-1 \Leftrightarrow$ there exists some integer $m = 0, \ldots, k$ such that

$x_m < b_m$ and $x_i = b_i$, $i = m+1, \ldots, k$

$q_0, q_1, q_2, \ldots$ sequence of denominators of convergents of α

every integer $N \geqslant 1$ has unique representation $N = \sum_{i=0}^{k} b_i q_i$

$0 \leqslant b_0 < a_1$, $0 < b_k \leqslant a_{k+1}$, and $0 \leqslant b_i \leqslant a_{i+1}$, $i = 1, \ldots, n-1$

$b_{i-1} = 0$ if $b_i = a_{i+1}$, $i = 1, \ldots, n$

$q = \sum_{i=0}^{k} x_i q_i$ and $N = \sum_{i=0}^{k} b_i q_i$ $x_k \ldots x_m \ldots x_1 x_0 < b_k \ldots b_m \ldots b_1 b_0$

$0 \leqslant q \leqslant N-1 \Leftrightarrow$ there exists some integer $m = 0, \ldots, k$ such that

$x_m < b_m$ and $x_i = b_i$, $i = m+1, \ldots, k$

convergents $\frac{p_i}{q_i}$ of α, $\eta_i = q_i\alpha - p_i \begin{cases} > 0 & \text{if } i \text{ even} \\ < 0 & \text{if } i \text{ odd} \end{cases}$

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$\beta \in (0, 1-\alpha)$ if and only if $\min\{i = 0, 1, 2, 3, \ldots : c_i \geqslant 1\}$ is even

$$q = \sum_{i=0}^{k} x_i q_i$$

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$x_m < b_m$ exclude N where $b_i = a_{i+1}$ for some $i = 1,\ldots,k$

$x_i = b_i,\ i = m+1,\ldots,k$ ε proportion among $N \in [0, q_{k+1})$

$\beta = \sum_{i=0}^{\infty} c_i \eta_i$ with c_i all even, $c_i \neq 0$ if i even

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contributions to $\Phi(\alpha; \beta; N)$:

∘ $\mathcal{I}_1$: $m > \ell$ with ℓ even

∘ $\mathcal{I}_2$: $m > \ell$ with ℓ odd

∘ $\mathcal{I}_3$: $m = \ell$ with ℓ even

∘ $\mathcal{I}_4$: $m = \ell$ with ℓ odd

∘ $\mathcal{I}_5$: $\ell > m$

$\beta = \sum_{i=0}^{\infty} c_i \eta_i$ with c_i all even, $c_i \neq 0$ if i even

$N = \sum_{i=0}^{k} b_i q_i$ subject to some exclusion

contributions to $\Phi(\alpha; \beta; N)$:

- $\mathcal{I}_1$: $m > \ell$ with ℓ even — $\mathcal{E}_1$: error for $\ell = 0$
- $\mathcal{I}_2$: $m > \ell$ with ℓ odd
- $\mathcal{I}_3$: $m = \ell$ with ℓ even — $\mathcal{E}_3$: error for $\ell = 0$
- $\mathcal{I}_4$: $m = \ell$ with ℓ odd
- $\mathcal{I}_5$: $\ell > m$

$$\Phi(\alpha;\beta;N) = \sum_{\ell=0}^{k} \min\{b_\ell, c_\ell\} + \sum_{\ell=1}^{k} \Delta_\ell + \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_3 \bmod 2$$

for $1-\varepsilon$ proportion among $N \in [0, q_{k+1})$

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for $1-\varepsilon$ proportion among $N \in [0, q_{k+1})$

$\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_3$ do not depend on sequence $c_0, c_1, c_2, c_3, \ldots$

$$\Phi(\alpha;\beta;N) = \sum_{\ell=0}^{k} \min\{b_\ell, c_\ell\} + \sum_{\ell=1}^{k} \Delta_\ell + \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_3 \bmod 2$$

for $1-\varepsilon$ proportion among $N \in [0, q_{k+1})$

$\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_3$ do not depend on sequence $c_0, c_1, c_2, c_3, \ldots$

$\beta' = \sum_{i=0}^{\infty} c_i' \eta_i$ and $\beta'' = \sum_{i=0}^{\infty} c_i'' \eta_i$

$\Phi(\alpha;\beta'';N) - \Phi(\alpha;\beta';N)$

$$\Phi(\alpha;\beta;N) = \sum_{\ell=0}^{k} \min\{b_\ell, c_\ell\} + \sum_{\ell=1}^{k} \Delta_\ell + \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_3 \bmod 2$$

for $1-\varepsilon$ proportion among $N \in [0, q_{k+1})$

$\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_3$ do not depend on sequence $c_0, c_1, c_2, c_3, \ldots$

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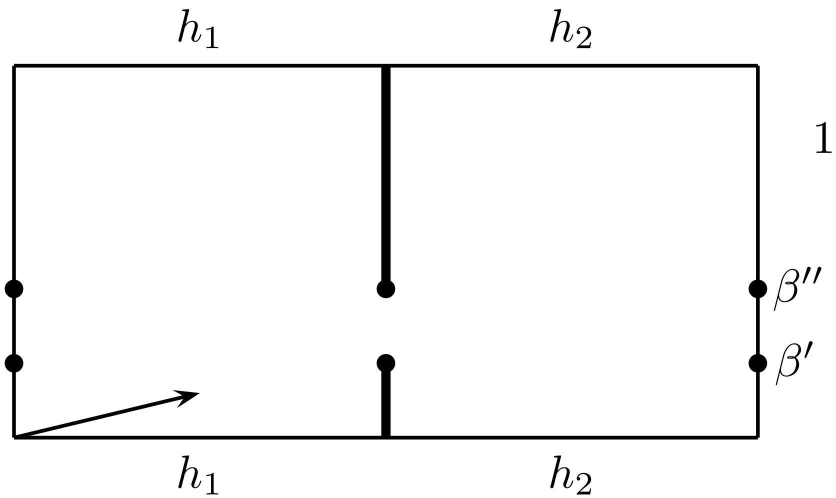
$\Phi(\alpha;\beta'';N) - \Phi(\alpha;\beta';N)$

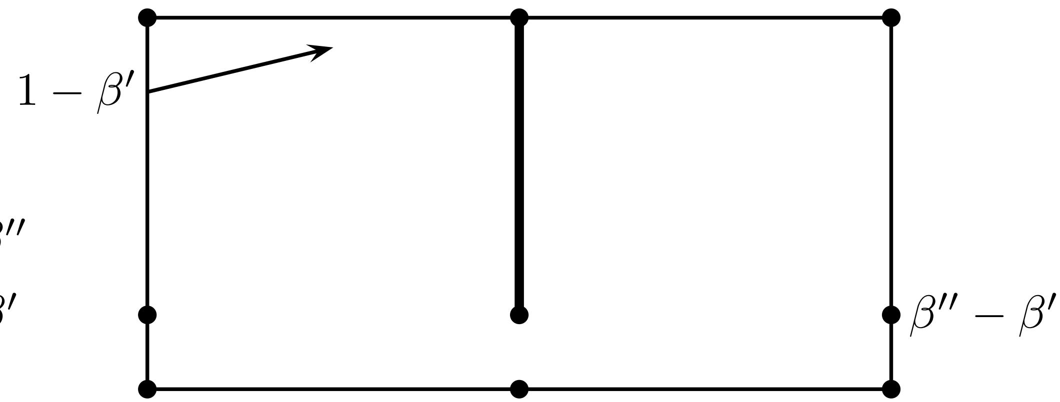
$$= \sum_{\ell=0}^{k} \min\{b_\ell, c_\ell'\} + \sum_{\ell=0}^{k} \min\{b_\ell, c_\ell''\} + \sum_{\ell=1}^{k} \Delta_\ell' + \sum_{\ell=1}^{k} \Delta_\ell'' \bmod 2$$

for $1-\varepsilon$ proportion among $N \in [0, q_{k+1})$

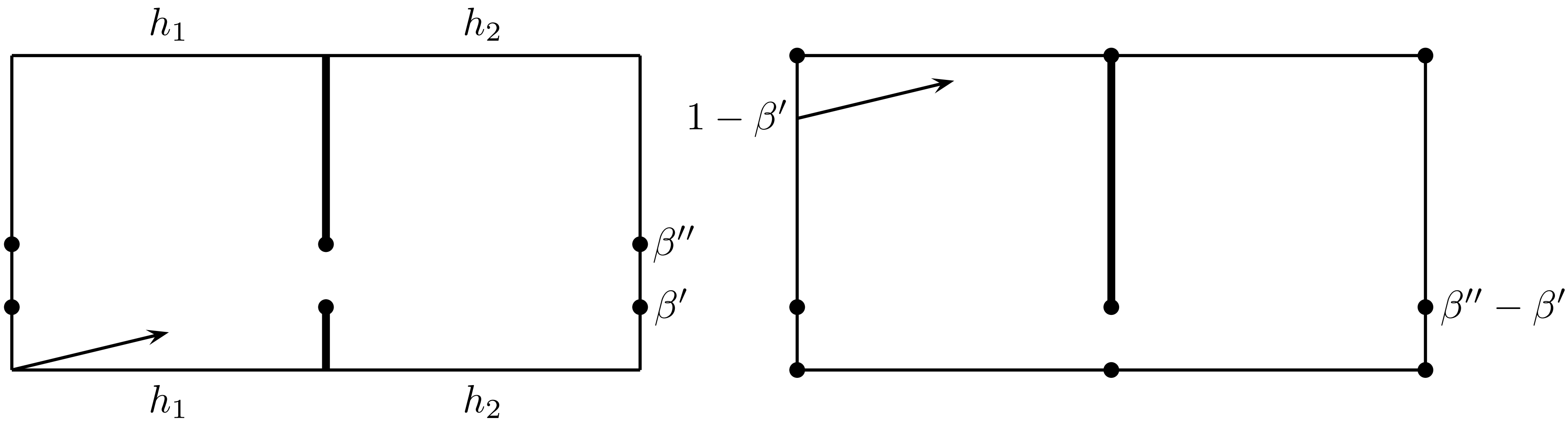
$\Phi(\alpha;\beta'';N) - \Phi(\alpha;\beta';N)$

$\Phi(\alpha; \beta''; N) - \Phi(\alpha; \beta'; N)$



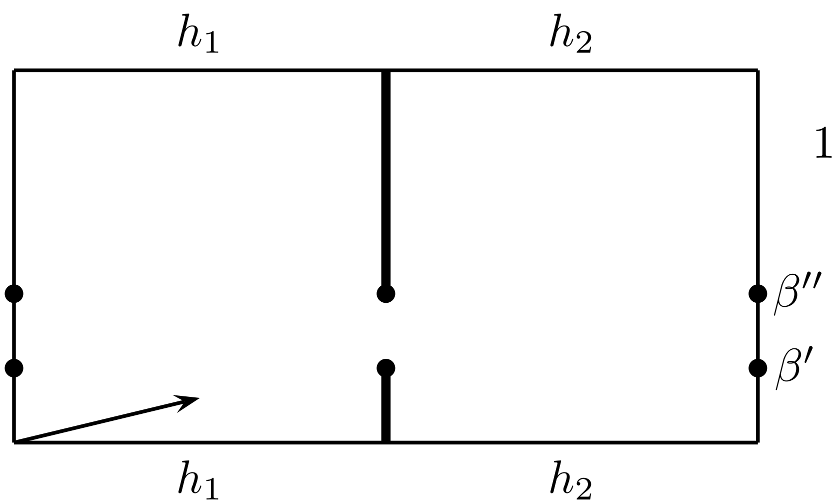


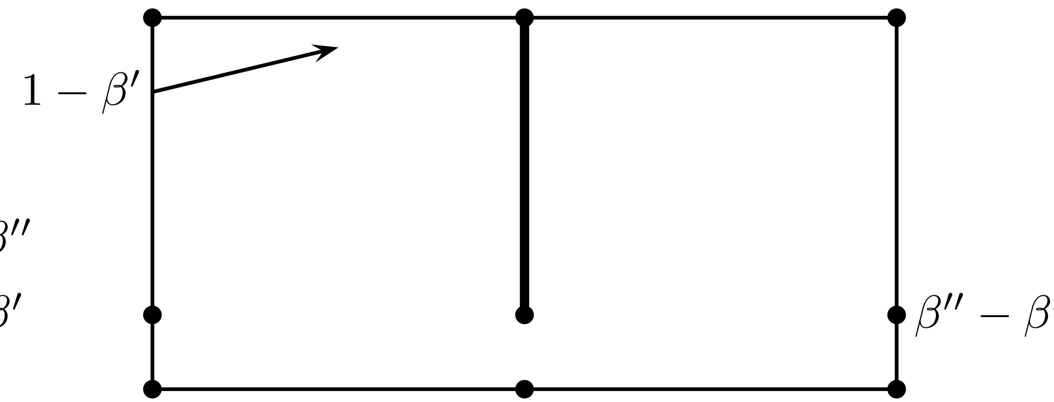
$\Phi(\alpha;\beta'';N)-\Phi(\alpha;\beta';N)$



$\beta_0'\ :\ c_i' = 2$

$$\beta_0''\ :\ c_i'' = \begin{cases} 4 & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd} \end{cases}$$





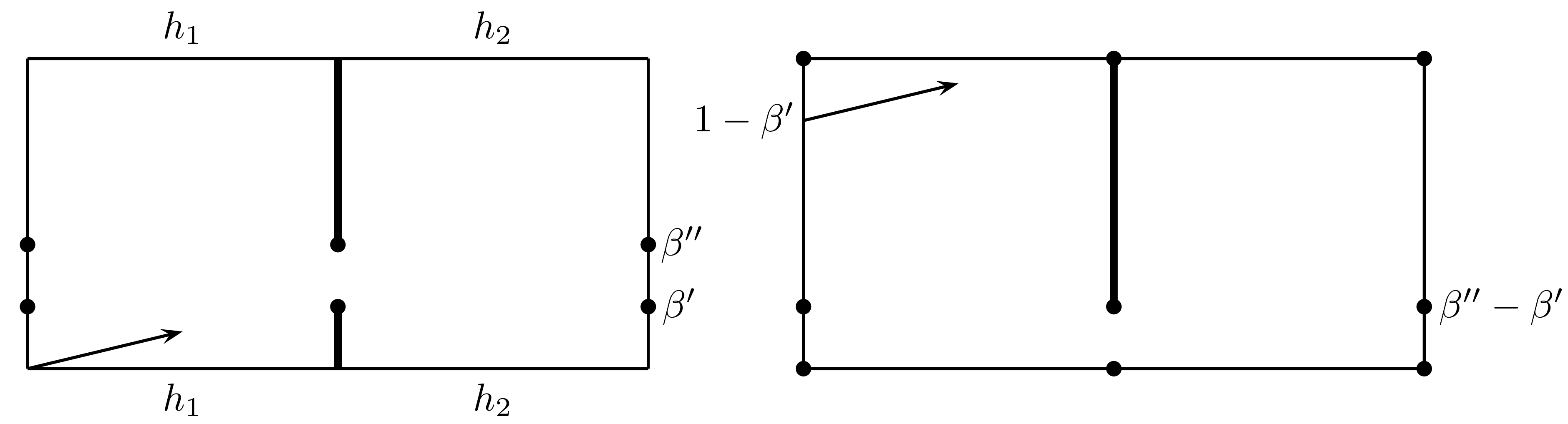
$\beta'_0 : c'_i = 2$

$$\beta'_1 : c'_i = \begin{cases} 2 & \text{if } i \text{ even} \\ 2 & \text{if } i \text{ odd and } i < 2n+2 \\ 0 & \text{if } i \text{ odd and } i > 2n+2 \end{cases}$$

$$\beta''_0 : c''_i = \begin{cases} 4 & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd} \end{cases}$$

$$\beta''_1 : c''_i = \begin{cases} 4 & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd} \end{cases}$$

$\Phi(\alpha; \beta''; N) - \Phi(\alpha; \beta'; N)$



$\beta'_0 \, : \, c'_i = 2$

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$$\beta''_0 \, : \, c''_i = \begin{cases} 4 & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd} \end{cases}$$

$$\beta''_1 \, : \, c''_i = \begin{cases} 4 & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd} \end{cases}$$

THANK YOU