Estimates for the discrete energies on the sphere

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Let K_d^ψ be the positive definite function

$$K_d^\psi(t) = \sum_{\ell=0}^{\infty} \psi(\ell) Z(d,\ell) P_\ell^{(d)}(t), \quad \psi(\ell) \geq 0, \tag{1}$$

where $P_\ell^{(d)}$ is the ℓ-th generalized Legendre polynomial.
For a set of N points X_N on the sphere $\mathbb{S}^d$ we consider the discrete energy

$$E(K_d^\psi, X_N) := \frac{1}{N^2} \sum_{k,j=1}^{N} K_d^\psi(\langle \mathbf{x}_k, \mathbf{x}_j \rangle) \tag{2}$$

and the minimal discrete energy of N points on the sphere

$$\mathcal{E}(K_d^\psi, N) := \inf_{X_N} E(K_d^\psi, X_N). \tag{3}$$

The worst-case error of numerical integration

Definition

The worst-case (cubature) error of the cubature rule $Q[X_N, \omega]$ in a Banach space B of continuous functions on $\mathbb{S}^d$ with norm $\|\cdot\|_B$ is defined by

$$\mathrm{wce}(Q[X_N, \omega]; B) := \sup_{f \in B, \|f\|_B \leq 1} |Q[X_N, \omega](f) - \mathrm{I}(f)|, \qquad (4)$$

where

$$\mathrm{I}(f) := \int_{\mathbb{S}^d} f(\mathbf{x}) d\sigma_d(\mathbf{x}),$$

$$Q[X_N, \omega](f) := \sum_{i=1}^{N} \omega_i f(\mathbf{x}_i), \qquad \sum_{j=1}^{N} \omega_j = 1.$$

By $Q[X_N](f)$ we will denote the equal weight numerical integration rule.

The worst case error in a reproducing kernel Hilbert space B can be represented

$$\mathrm{wce}(Q[X_N,\omega];B)^2 = \sum_{i,j=1}^{N} \omega_i \omega_j \tilde{K}_B(\langle \mathbf{x}_i, \mathbf{x}_j \rangle), \tag{5}$$

where K_B is a reproducing kernel.

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(ii) $K_B(\cdot, \mathbf{x}) \in B$ for all fixed $\mathbf{x} \in B$; and

(iii) the reproducing property

$$(f, K_B(\cdot, \mathbf{x}))_B = f(\mathbf{x}) \quad \forall f \in B \quad \forall \mathbf{x} \in \mathbb{S}^d.$$

Indeed

$$\mathrm{wce}(Q[X_N,\omega];B)^2 =$$

$$= \sup_{f\in B,\ \|f\|_B\le 1} \left| \left(f,\ \sum_{j=1}^{N} \omega_j K_B(\cdot,\mathbf{x}_j) - \int_{\mathbb{S}^d} K_B(\cdot,\mathbf{y}) d\sigma_d(\mathbf{y}) \right) \right|^2$$

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$$= \sum_{j=1}^{N} \omega_j \sum_{i=1}^{N} \omega_i K_B(\mathbf{x}_i,\mathbf{x}_j) - 2\sum_{i=1}^{N} \omega_i \int_{\mathbb{S}^d} K_B(\mathbf{x}_i,\mathbf{y})d\sigma_d(\mathbf{y})$$

$$+ \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} K_B(\mathbf{x},\mathbf{y})d\sigma_d(\mathbf{x})d\sigma_d(\mathbf{y})$$

$$= \sum_{j=1}^{N} \omega_j \sum_{i=1}^{N} \omega_i K_B(\mathbf{x}_i, \mathbf{x}_j) - \int_{\mathbb{S}^d} K_B(\mathbf{x}, \mathbf{y}) d\sigma_d(\mathbf{y}),$$

$$= \sum_{j=1}^{N} \omega_j \sum_{i=1}^{N} \omega_i \tilde{K}_B(\mathbf{x}_i, \mathbf{x}_j),$$

where we have used the reproducing property of K_B.

We denote by $\{Y_{\ell,k}^{(d)} : k = 1, \ldots, Z(d,\ell)\}$ a collection of $\mathbb{L}_2$-orthonormal real spherical harmonics (homogeneous harmonic polynomials in $d+1$ variables restricted to $\mathbb{S}^d$) of degree ℓ, where

$$Z(d,0) = 1, \quad Z(d,\ell) = (2\ell + d - 1)\frac{\Gamma(\ell + d - 1)}{\Gamma(d)\Gamma(\ell + 1)}. \qquad (6)$$

Each spherical harmonic $Y_{\ell,k}^{(d)}$ of exact degree ℓ is an eigenfunction of the negative Laplace-Beltrami operator $-\Delta_d^*$ with eigenvalue

$$\lambda_\ell := \ell(\ell + d - 1).$$

The Sobolev space $\mathbb{H}^s(\mathbb{S}^d)$ for $s \geq 0$ consists of all functions $f \in \mathbb{L}_2(\mathbb{S}^d)$ with finite norm

$$\|f\|_{\mathbb{H}^s} = \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(d,\ell)} (1+\lambda_\ell)^s \, |\hat{f}_{\ell,k}|^2 \right)^{\frac{1}{2}}, \qquad (7)$$

where the Laplace-Fourier coefficients are given by the formula

$$\hat{f}_{\ell,k} := (f, Y_{\ell,k}^{(d)})_{\mathbb{S}^d} = \int_{\mathbb{S}^d} f(\mathbf{x}) Y_{\ell,k}^{(d)}(\mathbf{x}) d\sigma_d(\mathbf{x}).$$

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The reproducing kernel $K_{d,s}$ in $\mathbb{H}^s(\mathbb{S}^d)$ is given by

$$K_{d,s}(t) = \sum_{\ell=0}^{\infty} (1+\lambda_\ell)^{-s} \, Z(d,\ell) P_\ell^{(d)}(t). \tag{8}$$

The space $\mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d)$ will be defined for $\gamma > \frac{1}{2}$, $d \geq 2$, as the set of all functions $f \in \mathbb{L}_2(\mathbb{S}^d)$ whose Laplace-Fourier coefficients satisfy

$$\|f\|^2_{\mathbb{H}^{(\frac{d}{2},\gamma)}} := \sum_{\ell=0}^{\infty} (1+\lambda_\ell)^{\frac{d}{2}} \left(\ln\left(3+\lambda_\ell\right)\right)^{2\gamma} \sum_{k=1}^{Z(d,\ell)} |\hat{f}_{\ell,k}|^2 < \infty. \quad (9)$$

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The reproducing kernel $K_{d,\gamma}$ in $\mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d)$ is given by

$$K_{d,\gamma}(t) = \sum_{\ell=0}^{\infty} (1+\lambda_\ell)^{-\frac{d}{2}} (\ln(3+\lambda_\ell))^{-2\gamma} Z(d,\ell) P_\ell^{(d)}(t). \quad (10)$$

The generalized Sobolev space $\mathbb{H}^{\psi}(\mathbb{S}^d)$ consists of all functions $f \in \mathbb{L}_2(\mathbb{S}^d)$ with finite norm

$$\|f\|_{\mathbb{H}^{\psi}} = \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(d,\ell)} \frac{1}{\psi(\ell)} |\hat{f}_{\ell,k}|^2 \right)^{\frac{1}{2}}. \qquad (11)$$

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The reproducing kernel K_d^{ψ} of the Hilbert space $\mathbb{H}^{\psi}(\mathbb{S}^d)$ is given by

$$K_d^{\psi}(t) = \sum_{\ell=0}^{\infty} \psi(\ell) Z(d,\ell) P_{\ell}^{(d)}(t). \qquad (12)$$

$\mathbb{H}^\psi(\mathbb{S}^d)$ is a reproducing kernel Hilbert space. A kernel K_d^ψ : $\mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ has the following properties:
(i) $K_d^\psi(\mathbf{x},\mathbf{y}) = K_d^\psi(\mathbf{y},\mathbf{x})$ for all $\mathbf{x},\mathbf{y} \in \mathbb{S}^d$;
(ii) $K_d^\psi(\cdot,\mathbf{x}) \in \mathbb{H}^\psi(\mathbb{S}^d)$ for all fixed $\mathbf{x} \in \mathbb{H}^\psi(\mathbb{S}^d)$; and
(iii) the reproducing property

$$(f, K_d^\psi(\cdot,\mathbf{x}))_{\mathbb{H}^\psi} = f(\mathbf{x}) \quad \forall f \in \mathbb{H}^\psi(\mathbb{S}^d) \quad \forall \mathbf{x} \in \mathbb{S}^d.$$

Spherical t-designs

Definition A spherical t-design on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is a set

$$X_N := \{\mathbf{x}_1, ..., \mathbf{x}_N\} \in \mathbb{S}^d,$$

such that for all polynomials p with degree $\leq t$

$$\frac{1}{N} \sum_{\mathbf{x} \in X_N} p(\mathbf{x}) = \int_{\mathbb{S}^d} p(\mathbf{x}) d\sigma_d(\mathbf{x}), \quad \deg(p) \leq t.$$

Here N is the number of points of spherical design.

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Here N is the number of points of spherical design.

Definition A set $X_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ is called well-separated if there exists a positive constant c_1 such that

$$\min_{i \neq j} |\mathbf{x}_i - \mathbf{x}_j| \geq \frac{c_1}{N^{\frac{1}{d}}}. \tag{13}$$

The definition of spherical t-design is equivalent to

$$\sum_{(\mathbf{x},\mathbf{y})\in X_N\times X_N} P_\ell^{(d)}(\langle \mathbf{x},\mathbf{y}\rangle)=0, \qquad (14)$$

for $\ell=1,\ldots,t$

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for $\ell = 1, ..., t$
It is an immediate consequence of the fact that the restrictions of polynomials to $\mathbb{S}^d$ are spanned by the harmonic polynomials and the addition theorem for spherical harmonics

$$\sum_{k=1}^{Z(d,\ell)} Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}) = Z(d,\ell) P_\ell^{(d)}(\langle \mathbf{x},\mathbf{y}\rangle).$$

Relation between t and N

Delsarte, Goethals and Seidel'77 established lower bounds

$$N \geq \begin{cases} \binom{d+t/2}{d} + \binom{d+t/2-1}{d} & \text{for } t \text{ even },\\ 2\binom{d+\lfloor t/2 \rfloor}{d} & \text{for } t \text{ odd },\end{cases}$$

but these lower bounds can be achieved only for a few small values of t.

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Seymour and Zaslavsky'84 showed that for every $t \geq 1$ (and for every dimension of the sphere) there always exists a spherical design.

Korevaar and Meyers'93 conjectured that there always exist spherical t-designs with $N \asymp t^d$ points.

Theorem (Bondarenko, Radchenko and Viazovska'13)

For $d \geq 2$, there exists a constant c_d depending only on d such that for every $N \geq c_d t^d$ there exists a spherical t-design with N points.

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Theorem (Bondarenko, Radchenko and Viazovska '15)

For each $d \geq 2$, $t \in \mathbb{N}$, $N > c_d t^d$, there exist positive constants c_d and λ_d depending only on d such that for every $N \geq c_d t^d$ there exists a spherical t-design on $\mathbb{S}^d$ consisting of N points $\{\mathbf{x}\}_{i=1}^N$ with $|\mathbf{x}_i - \mathbf{x}_j| \geq \lambda_d N^{-\frac{1}{d}}$ for $i \neq j$.

Theorem (Hesse and Sloan'05 ($s = \frac{3}{2}$, $d = 2$); Hesse and Sloan'06 ($s > 1$, $d = 2$), Brauchart and Hesse'07 ($s > \frac{d}{2}$, $d \geq 2$))

Given $s > \frac{d}{2}$, there exists $C_{s,d} > 0$, such that for every N-point spherical t-design X_N on $\mathbb{S}^d$ there holds

$$\mathrm{wce}(Q[X_N]; \mathbb{H}^s(\mathbb{S}^d)) \leq \frac{C_{s,d}}{t^s}. \tag{15}$$

Theorem (Hesse and Sloan'05 ($d = 2$); , Hesse'06 ($d \geq 2$))

Given $s > \frac{d}{2}$, there exists $C_{s,d} > 0$, such that for every N-point configuration X_N on $\mathbb{S}^d$ there holds

$$\frac{C_{s,d}}{N^{\frac{s}{d}}} \leq \mathrm{wce}(Q[X_N, \omega]; \mathbb{H}^s(\mathbb{S}^d)). \tag{16}$$

Theorem (Grabner and S.'18)

Let $d \geq 2$, $\gamma > \dfrac{1}{2}$ be fixed, $X_{N(t)}$ be a well-separated spherical t-design and $N \asymp t^d$. Then there exist positive constants $C^{(1)}_{d,\gamma}$ and $C^{(2)}_{d,\gamma}$, such that

$$C^{(1)}_{d,\gamma} N^{-\frac{1}{2}} (\ln N)^{-\gamma+\frac{1}{2}} \leq \mathrm{wce}(Q[X_N]; \mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d)) \leq C^{(2)}_{d,\gamma} N^{-\frac{1}{2}} (\ln N)^{-\gamma+\frac{1}{2}}. \quad (17)$$

The constants $C^{(1)}_{d,\gamma}$ and $C^{(2)}_{d,\gamma}$ depend only on d and γ.

Theorem (Grabner and S.'18)

Let $d \geq 2$, $\gamma > \frac{1}{2}$, $Q[X_N, \omega]$ is an arbitrary N-point cubature rule. Then, there exists a positive constant $C_{d,\gamma}$ such that

$$\mathrm{wce}(Q[X_N, \omega]; \mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d)) \geq C_{d,\gamma} N^{-\frac{1}{2}} (\ln N)^{-\gamma} . \qquad (18)$$

The constant $C_{d,\gamma}$ depends only on d and γ, but is independent of the rule $Q[X_N, \omega]$ and the number of nodes N of the rule.

As consequence, we get

$$1 + \frac{C^{(1)}_{s,d}}{N^{\frac{s}{d}}} \le \mathcal{E}(K_{d,s}, N) \le 1 + \frac{C^{(2)}_{s,d}}{N^{\frac{s}{d}}}. \tag{19}$$

As consequence, we get

$$1 + \frac{C_{s,d}^{(1)}}{N^{\frac{s}{d}}} \leq \mathcal{E}(K_{d,s}, N) \leq 1 + \frac{C_{s,d}^{(2)}}{N^{\frac{s}{d}}}. \tag{19}$$

$$(\ln 3)^{2\gamma} + C_{d,\gamma}^{(1)} N^{-\frac{1}{2}} (\ln N)^{-\gamma} \leq \mathcal{E}(K_{d,\gamma}, N) \leq$$

$$\leq (\ln 3)^{2\gamma} + C_{d,\gamma}^{(2)} N^{-\frac{1}{2}} (\ln N)^{-\gamma + \frac{1}{2}}. \tag{20}$$

By Θ_d,we denote the set of monotonically nonincreasing functions $\psi(t)$ for which there exists a constant $\alpha > d$ such that the function $t^\alpha \psi(t)$ almost decreases, i.e., there exists a positive constant K such that $t_1^\alpha \psi(t_1) \leq K t_2^\alpha \psi(t_2)$ for any $1 \leq t_2 \leq t_1$.

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By B we denote the set of positive functions $\psi(t)$ monotonically nonincreasing for $t \ge 1$ for which there exists a positive constant K such that

$$\frac{\psi(t)}{\psi(2t)} \le K \quad \forall t \ge 1. \tag{21}$$

Examples of the functions ψ, satisfying the condition $\psi \in B \cap \Theta_d$ are the functions of the form:

$$\psi(t) = (t+1)^{-s}, \quad s > d; \tag{22}$$

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$$\psi(t) = \frac{\ln^\alpha(t+c)}{(t+1)^s}, \; s > d, \quad \alpha \geq 0, \quad c > e^{\frac{2\alpha}{s-d}} - 1. \tag{24}$$

Theorem (S.'20)

Let $d \geq 2$ be fixed and $\psi \in B \cap \Theta_d$. Then

$$\psi(0) + C^{(1)}_{d,\psi}\psi\left(N^{\frac{1}{d}}\right) \leq \mathcal{E}(K^{\psi}_{d}, N) \leq \psi(0) + C^{(2)}_{d,\psi}\psi\left(N^{\frac{1}{d}}\right). \quad (25)$$

The constants $C^{(1)}_{d,\psi}$ and $C^{(2)}_{d,\psi}$ depend only on d and ψ.

Lemma 1 (S.'20)

Let $d \geq 2$ be fixed, $\psi \in B \cap \Theta_d$ and X_N be N point well-separated spherical t-design with $N \asymp t^d$. Then there exists a positive constant $C_{d,\psi}$ such that

$$E(K_d^{\psi}, X_N) - \psi(0) \leq C_{d,\psi}\psi\left(N^{\frac{1}{d}}\right). \tag{26}$$

The constant $C_{d,\psi}$ depends only on d and ψ.

Lemma 2 (S.'20)

Let $d \geq 2$ be fixed, and $\psi \in B \cap \Theta_d$, X_N be an arbitrary N-point sequence. Then, there exists a positive constant $C_{d,\psi}$ such that

$$E(K_d^{\psi}, X_N) - \psi(0) \geq C_{d,\psi} \psi \left(N^{\frac{1}{d}} \right). \quad (27)$$

The constant $C_{d,\psi}$ depends only on d and ψ, but is independent of the number of nodes N.

Corollary

Let $d \geq 2$ be fixed and
$\psi_1(k) = \frac{1}{(k+1)^s \ln^\alpha (k+c)}$, $s > d$, $\alpha \geq 0$, $c > 1$. Then

$$\ln^{-\alpha} c + \frac{C^{(1)}_{d,\psi_1}}{N^{\frac{s}{d}} \ln^\alpha N} \leq \mathcal{E}(K_d^{\psi_1}, N) \leq \ln^{-\alpha} c + \frac{C^{(1)}_{d,\psi_1}}{N^{\frac{s}{d}} \ln^\alpha N}. \qquad (28)$$

The constants $C^{(1)}_{d,\psi_1}$ and $C^{(2)}_{d,\psi_1}$ depend only on d and ψ_1.

Let K_d be the positive definite function

$$K_d(t) := \sum_{n=0}^{\infty} a_n P_n^{(d)}(t), \quad a_n \geq 0. \tag{29}$$

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For $X_N \subset \mathbb{S}^d$, $X_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ let consider energies of the form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} K_d(\langle \mathbf{x}_i, \mathbf{x}_j \rangle), \tag{30}$$

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Let $\{A_i\}_{i=1}^N$ be an area regular partition of the sphere, i.e.:
$\mathbb{S}^d = \bigcup\limits_{i=1}^{N} A_i$, $A_i \cap A_j = \emptyset$, $i \neq j$ and $\sigma(A_i) = \frac{1}{N}$,
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$\mathrm{diam}(A_i) \leq C N^{-\frac{1}{d}}$ for $i = 1, ..., N$.
Choose the points $X_N = \{\mathbf{x}_1, ..., \mathbf{x}_N\}$ uniformly randomly in A_i:
$\mathbf{x}_i \in A_i$.

Let σ_j^* be the restriction of the measure $N\sigma$ to A_j: $\sigma_j^*(\cdot) = \sigma(A_j \cap \cdot)N$. Then each σ_j^* is a probability measure.

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$$\frac{1}{N^2}\int\limits_{A_1} \ldots \int\limits_{A_N} \sum_{i,j=1}^{N} K_d(\langle \mathbf{x}_i, \mathbf{x}_j \rangle) d\sigma_1^*(\mathbf{x}_1)\ldots d\sigma_N^*(\mathbf{x}_N), \quad K_d \in \mathbb{C}_{[-1,1]} \tag{31}$$

Theorem (Grabner and S.'18)

Let K_d be a continuous function on $[-1,1]$, which is defined by (29). Then there exists a positive constant C_d, such that the following estimate holds

$$\frac{1}{N^2}\int\limits_{A_1}\dots\int\limits_{A_N}\sum_{i,j=1}^{N}K_d(\langle \mathbf{x}_i,\mathbf{x}_j\rangle)d\sigma_1^*(\mathbf{x}_1)\dots d\sigma_N^*(\mathbf{x}_N)-a_0$$

$$\leq \frac{C_d}{N}\left(N^{-\frac{2}{d}}\sum_{n=1}^{[N^{\frac{1}{d}}]}a_n n^2+\sum_{n=[N^{\frac{1}{d}}]+1}^{\infty}a_n\right). \quad (32)$$

The constant C_d depends on d, but is independent on N.

For the reproducing kernel of Sobolev space $\mathbb{H}^s(\mathbb{S}^d)$,

$$\frac{1}{N^2}\int\limits_{A_1}\dots\int\limits_{A_N}\sum_{i,j=1}^{N} K_d^{(s)}(\langle \mathbf{x}_i, \mathbf{x}_j\rangle)d\sigma_1^*(\mathbf{x}_1)\dots d\sigma_N^*(\mathbf{x}_N) - 1$$

$$\ll \begin{cases} N^{-\frac{2s}{d}}, & \text{if } \frac{d}{2} < s < 1+\frac{d}{2}, \\ N^{-1-\frac{2}{d}}\ln N, & \text{if } s = 1+\frac{d}{2}, \\ N^{-1-\frac{2}{d}}, & \text{if } s > 1+\frac{d}{2}. \end{cases} \qquad (33)$$

For the reproducing kernel of the space $\mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d)$, $\gamma > \frac{1}{2}$,

$$\frac{1}{N^2}\int\limits_{A_1}\dots\int\limits_{A_N}\sum_{i,j=1}^{N}\tilde{K}^{(\frac{d}{2},\gamma)}(\langle \mathbf{x}_i,\mathbf{x}_j\rangle)d\sigma_1^*(\mathbf{x}_1)\dots d\sigma_N^*(\mathbf{x}_N) \ll N^{-1}(\ln N)^{-2\gamma+1}. \tag{34}$$

For the reproducing kernel of Sobolev space $\mathbb{H}^{\psi}(\mathbb{S}^d)$, the following estimate holds

$$\frac{1}{N^2}\int\limits_{A_1}\cdots\int\limits_{A_N}\sum_{i,j=1}^{N} K_d^{(\psi)}(\langle \mathbf{x}_i,\mathbf{x}_j\rangle)d\sigma_1^*(\mathbf{x}_1)\ldots d\sigma_N^*(\mathbf{x}_N)-\psi(0)$$

$$\ll \frac{1}{N}\left(N^{-\frac{2}{d}}\sum_{n=1}^{[N^{\frac{1}{d}}]}\psi(n)n^{d+1}+\sum_{n=[N^{\frac{1}{d}}]+1}^{\infty}\psi(n)n^{d-1}\right). \qquad (35)$$

So, for jittered sampling $\{X_N\}$ and for $\psi \in B \cap \Theta_d$

$$E(K_d^{\psi}, X_N) - \psi(0) \leq \frac{C_{d,\psi}}{N} \left(N^{-\frac{2}{d}} \sum_{n=1}^{[N^{\frac{1}{d}}]} \psi(n) n^{d+1} + N\psi\left(N^{\frac{1}{d}}\right) \right). \qquad (36)$$

So, for jittered sampling $\{X_N\}$ and for $\psi \in B \cap \Theta_d$

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In particular, if $\psi \in B \cap \Theta_{d+2}$

$$E(K_d^{\psi}, X_N) - \psi(0) \leq C_{d,\psi} N^{-1-\frac{2}{d}}. \tag{37}$$

So, for jittered sampling $\{X_N\}$ and for $\psi \in B \cap \Theta_d$

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$$E(K_d^\psi, X_N) - \psi(0) \leq C_{d,\psi} N^{-1-\frac{2}{d}}. \qquad (37)$$

and if $\psi \in B \cap (\Theta_d \setminus \Theta_{d+2-\varepsilon})$

$$E(K_d^\psi, X_N) - \psi(0) \leq C_{d,\psi} \psi\left(N^{\frac{1}{d}}\right). \qquad (38)$$

THANK YOU FOR YOUR ATTENTION!!!

References

D. Bilyk and F. Dai, Geodesic distance Riesz energy on the sphere, Trans. Amer. Math. Soc. 372 (2019), 3141-3166

A. Bondarenko, D. Radchenko, and M. Viazovska.
Optimal asymptotic bounds for spherical designs.
Ann. of Math. 178(2):443–452, 2013.

A. Bondarenko, D. Radchenko, and M. Viazovska.
Well-separated spherical designs.
Constr. Approx. 41(1):93–112, 2015.

J. S. Brauchart and K. Hesse.
Numerical integration over spheres of arbitrary dimension.
Constr. Approx. 25(1):41–71, 2007.

P. Delsarte, J. M. Goethals, and J. J. Seidel.
Spherical codes and designs.
Geometriae Dedicata 6(3):363–388, 1977.

K. Hesse, A lower bound for the worst-case cubature error on spheres of arbitrary dimension, Numer. Math. 103 (3) (2006) 413–433.

K. Hesse, I.H. Sloan, Optimal lower bounds for cubature error on the sphere S2, J. Complexity 21 (6) (2005) 790–803.

K. Hesse, I.H. Sloan, Cubature over the sphere S2 in Sobolev spaces of arbitrary order, J. Approx. Theory 141 (2) (2006) 118–133.