Uncertainty principles, interpolation formulas and packing problems

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This presentation is based on

I. Fourier uniqueness pairs of powers of integers (Joint with J.P.G. Ramos). Available at arXiv:1910.04276

II. Perturbed interpolation formulae and applications (Joint with J. P. G. Ramos). Available at arXiv:2005.10337

But some events may be changed for dramatic effect...

Remark

From now on, until the end of time, we use the normalization

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \xi} \mathrm{dx}$$

The three topics in the title

Packing problems

Try to fit as much as you can of a certain shape inside a bigger space with as little loss as possible of a certain quantity (Surface area of the boundaries, volume etc)

Interpolation formulas

Sometimes a class of functions is nice enough for you to determine it just by sampling the functions over "small" sets.

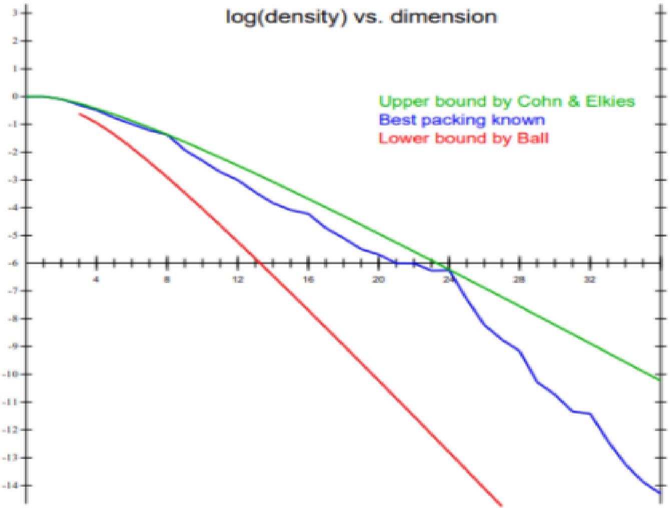
Uncertainty

Situations where knowing too much about different measurements of an object means it has to be very simple. In other words, you can not be certain about many measurements at the same time.

Sphere packing

Theorem (Cohn-Elkies-Gorbachev bounds)

If $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function and r is a positive scalar such that $f(0) = \widehat{f}(0) = 1$, $f(x) \geq 0$ for all x, and $\widehat{f}(u) \leq 0$ for $|u| \geq r$, then the density of a sphere packing in R^d is at most $vol(B_d(r/2))$



Sphere packing

Remark

If one wants to solve the sphere packing problem in a certain dimension via a lattice packing and the recently introduced bound, it follows from the proof of the Cohn-Elkies-Gorbachov result that to find a extremal radius one has to look for functions that are radial and are zero over the lattice except for the origin, and also have derivatives equal to zero over the lattice, except for the first crossing.

Remark

Since there are lower bounds in every dimension for the optimal packing density, one can conclude that the minimal radius on the Cohn-Elkies-Gorbachov theorem is in fact positive for every dimension.

Sphere packing

± 1-Sign uncertainty principle

There is $\rho = \rho_d(\pm 1)$ such that, if $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function and r is a positive scalar such that $f(0) = \widehat{f}(0) = 1$, $f(x) \geq 0$ for all x, and $(\pm 1)\widehat{f}(u) \geq 0$ for $|u| \geq$, then $r \geq \rho$.

Theorem (Cohn & Gonçalves - 2017)

Let $\mathcal{A}_d(\pm 1)$ represent the biggest $\rho_d(\pm 1)$ possible. Then

1. $\mathcal{A}_{12}(+1) = \sqrt{2}$
2. *The limits* $\lim_{d\to\infty} \dfrac{\mathcal{A}_d(+1)}{\sqrt{d}}$ *and* $\lim_{d\to\infty} \dfrac{\mathcal{A}_d(-1)}{\sqrt{d}}$ *exist and are equal.*

Remark

There is interesting work by Carneiro and Quesada from last month (arXiv:2006.00959) with new uncertainty principle where ± 1 is replaced by functions $P(x)$ and they also exhibit a mechanism that generates new sharp results from old ones by "jumping 4 dimensions".

Sphere packing

Theorem (Viazovska - 2016)

There exists a radial Schwartz function $f : \mathbb{R}^8 \to \mathbb{R}_+$ such that

(i) $f(0) = \widehat{f}(0) = 1$.

(ii) $\widehat{f}(u) \leq 0$ *for* $|u| \geq \sqrt{2}$.

In particular this means that the E_8 lattice solves the sphere packing problem for $\mathbb{R}^8$, and $\mathcal{A}_8(-1) = \sqrt{2}$.

Theorem (Cohn, Kumar, Miller, Radchenko, Viazovska - 2016)

There exists a radial Schwartz function $f : \mathbb{R}^{24} \to \mathbb{R}_+$ such that

(i) $f(0) = \widehat{f}(0) = 1$.

(ii) $\widehat{f}(u) \geq 0$ *for* $|u| \geq 1$.

In particular this means that the Leech lattice solves the sphere packing problem for $\mathbb{R}^{24}$ and $\mathcal{A}_{24}(-1) = 1$.

Theorem (Radchenko-Viazovska - 2017)

There are functions $a_n \in \mathcal{S}(\mathbb{R})$ such that for any given even function $f : \mathbb{R} \to \mathbb{C}$ that belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$ one has the following identity

$$f(x) = \sum_{k=0}^{\infty} f(\sqrt{k}) a_n(x) + \sum_{k=0}^{\infty} \widehat{f}(\sqrt{k}) \widehat{a_n}(x),$$

where the right-hand side converges absolutely.

Theorem (Cohn, Kumar, Miller, Radchenko, Viazovska - 2019)

Let (d, n_0) be either $(8,1)$ or $(24,2)$. There are functions $a_n, b_n \in \mathcal{S}(\mathbb{R})$ such that every $f \in \mathcal{S}_{rad}(\mathbb{R}^d)$ can be uniquely recovered by the sets of values

$$\{f(\sqrt{2n}), f'(\sqrt{2n}), \widehat{f}(\sqrt{2n}), \widehat{f}'(\sqrt{2n})\},\ n \geq n_0,$$

through the interpolation formula

$$\begin{aligned} f(x) = & \sum_{n \geq n_0} f(\sqrt{2n}) a_n(x) + \sum_{n \geq n_0} f'(\sqrt{2n}) b_n(x) \\ & + \sum_{n \geq n_0} \widehat{f}(\sqrt{2n}) \widehat{a_n}(x) + \sum_{n \geq n_0} \widehat{f}'(\sqrt{2n}) \widehat{b_n}(x), \end{aligned}$$

where the right-hand side converges absolutely.

Theorem (Shannon-Whittaker Interpolation - 1898)

Let f be a function in $L^2(\mathbb{R})$ whose Fourier transform $\widehat{f}$ is supported on the interval $[-1/2, 1/2]$. Then

$$f(x) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(x-k)}{\pi(x-k)},$$

where convergence holds uniformly and in the $L^2(\mathbb{R})$ sense.

Theorem (Vaaler Interpolation - 1985)

Let $f \in L^2(\mathbb{R})$, and suppose that $\widehat{f}$ is supported on $[-1,1]$. Then

$$f(x) = \frac{\sin^2(\pi x)}{\pi^2} \sum_{k \in \mathbb{Z}} \left\{ \frac{f(k)}{(x-k)^2} + \frac{f'(k)}{x-k} \right\}. \tag{1}$$

where convergence holds uniformly and in the $L^2(\mathbb{R})$ sense.

Question

1. The Shannon-Whittaker theorem actually implies Vaaler's result (Ramos-Sousa 2020). This begs the question: Does some sorte of interpolation without derivatives such as the Radchenko-Viazovska result imply something with derivatives and half the interpolation points?

2. The Shannon-Whittaker result admits perturbations, namely, Kadec-1/4 theorem. Does the Radchenko-Viazovska interpolation admits interpolation?

3. Vaaler's result also admits perturbations. Does the CKMRV interpolation also admits pertubations?

Perturbations

Theorem (Ramos - S. 2020)

There is $\delta > 0$ so that, for each sequence of real numbers $\{\varepsilon_k\}_{k\geq 0}$ such that $\varepsilon_k \in (-1/2, 1/2), \varepsilon_0 = 0$, $\sup_{k\geq 0} |\varepsilon_k|(1+k)^{5/4} < \delta \, \forall \, k \geq 0$, there are sequences of functions $\{\theta_j\}_{j\geq 0}, \{\eta_j\}_{j\geq 0}$ with

$$|\theta_j(x)| + |\eta_j(x)| + |\widehat{\theta}_j(x)| + |\widehat{\eta}_j(x)| \lesssim (1+j)^{\mathcal{O}(1)}(1+|x|)^{-10}$$

such that for all $f \in \mathcal{S}_{even}(\mathbb{R})$

$$f(x) = \sum_{j\geq 0} \left(f(\sqrt{j+\varepsilon_j})\theta_j(x) + \widehat{f}(\sqrt{j+\varepsilon_j})\eta_j(x) \right).$$

Theorem (Ramos - S. 2020)

Let $b_n^{\pm} = a_n \pm \widehat{a_n}$, where $\{a_n\}_{n\geq 0}$ are the basis functions in the Radchenko-Viazovska interpolation. Then there is $c > 0$ such that for all positive integers $n \in \mathbb{N}$

$$|b_n^{\pm}(x)| \lesssim n^{1/4} \log^{3/2}(1+n) e^{-c\frac{|x|}{\sqrt{n}}},$$

$$|(b_n^{\pm})'(x)| \lesssim n^{3/4} \log^{3/2}(1+n) e^{-c\frac{|x|}{\sqrt{n}}}.$$

Perturbations

Theorem (Ramos - S. 2020)

There is $C_0 > 0$ so that the following holds: there is $\delta > 0$ so that, for each sequence ε_k so that $|\varepsilon_k| < \delta k^{-C_0}$, then any function $f \in \mathcal{S}_{rad}(\mathbb{R}^d)$ is uniquely determined by the values

$$\left(f(\sqrt{2n+\varepsilon_n}), f'(\sqrt{2n+\varepsilon_n}), \widehat{f}(\sqrt{2n+\varepsilon_n}), \widehat{f}'(\sqrt{2n+\varepsilon_n})\right)_{n \geq n_0}, \qquad (2)$$

where we let $(d, n_0) = (8,1)$ or $(24,2)$.

Theorem (Ramos - S. 2020)

Let $a_n, \widehat{a_n}, b_n, \widehat{b_n}$ be as in the CKMRV interpolation formula for $(d, n_0) = (8,1)$ or $(24,2)$. indeed, there is $\tau > 0$ so that

$$\sup_{l \in \{0,1,2\}} \sup_{x \in \mathbb{R}^d} (1+|x|)^{100} \left(|a_n^{(l)}(x)| + |\widehat{a_n}^{(l)}(x)| + |b_n^{(l)}(x)| + |\widehat{b_n}^{(l)}(x)|\right) \lesssim n^\tau, \quad (3)$$

A little about the proofs

Consider a sequence of ε_k such that $|\varepsilon_k| < \frac{1}{2}$. Given an even function $f \in \mathcal{S}(\mathbb{R})$, by the Radchenko–Viazovska interpolation result, we have the following equations

$$f\left(\sqrt{k+\varepsilon_k}\right) = \sum_{n\geq 0} f\left(\sqrt{n}\right) a_n\left(\sqrt{k+\varepsilon_k}\right) + \sum_{n\geq 0} \widehat{f}\left(\sqrt{n}\right) \widehat{a_n}\left(\sqrt{k+\varepsilon_k}\right)$$

$$\widehat{f}\left(\sqrt{k+\varepsilon_k}\right) = \sum_{n\geq 0} f\left(\sqrt{n}\right) \widehat{a_n}\left(\sqrt{k+\varepsilon_k}\right) + \sum_{n\geq 0} \widehat{f}\left(\sqrt{n}\right) a_n\left(\sqrt{k+\varepsilon_k}\right)$$

So this leads to naturally consider the following operator defined on $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$: $A(\mathbf{x},\mathbf{y}) = (A_1(\mathbf{x},\mathbf{y}), A_2(\mathbf{x},\mathbf{y}))$ given by

$$A_1(\mathbf{x},\mathbf{y})_k = \sum_{n\geq 0} x_n \cdot \left(a_n\left(\sqrt{k+\varepsilon_k}\right) - \delta_{n,k}\right) + \sum_{n\geq 0} y_n \cdot \widehat{a_n}\left(\sqrt{k+\varepsilon_k}\right)$$

$$A_2(\mathbf{x},\mathbf{y})_k = \sum_{n\geq 0} x_n \cdot \widehat{a_n}\left(\sqrt{k+\varepsilon_k}\right) + \sum_{n\geq 0} y_n \cdot \left(a_n\left(\sqrt{k+\varepsilon_k}\right) - \delta_{n,k}\right)$$

Our result, in a way, is proving that A has small norm and therefore we can invert a certain operator.

Uniqueness pairs

Question

Given a collection $\mathcal{C}$ of functions $f : \mathbb{R} \to \mathcal{C}$, what conditions can we impose on two sets $A, \widehat{A} \subset \mathbb{R}$ to ensure that the only function $f \in \mathcal{C}$ such that $f(x) = 0$ for every $x \in A$ and $\widehat{f}(\xi) = 0$ for every $\xi \in \widehat{A}$ is the zero function?

Definition (Uniqueness pairs)

Given a collection $\mathcal{C}$ of functions $f : \mathbb{R} \to \mathcal{C}$, we say two sets $A, \widehat{A} \subset \mathbb{R}$ for a uniqueness pair for $\mathcal{C}$ if the only function $f \in \mathcal{C}$ such that $f(x) = 0$ for every $x \in A$ and $\widehat{f}(\xi) = 0$ for every $\xi \in \widehat{A}$ is the zero function.

Question

What uniqueness pairs can we think of? Clearly several of the interpolation results so far provide uniqueness pairs. What else?

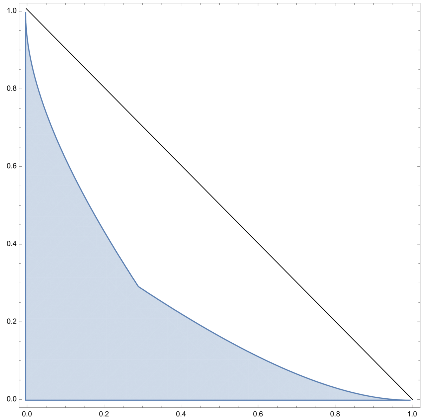
Uniqueness pairs

Theorem (Ramos - S. 2019)

Let $0 < \alpha, \beta < 1$ and $f \in \mathcal{S}(\mathbb{R})$. Then

(A) *If $f(\pm \log(n+1))) = 0$ and $\widehat{f}(\pm n^{\alpha}) = 0$ for every $n \in \mathbb{N}$, then $f \equiv 0$.*

(B) *There is a set A such that when $(\alpha, \beta) \in A$ and $f(\pm n^{\alpha}) = 0$ and $\widehat{f}(\pm n^{\beta}) = 0$ for every $n \in \mathbb{N}$, then $f \equiv 0$.*



Uniqueness pairs

Ideas

(i) Use the zeros to obtain decay: for $|x| > (2k+1)^\alpha$ and $|\xi| > (2j+1)^\beta$

$$|f(x)| \leq (k+1)!(2^{2-\alpha}\pi)^{k+1}\alpha^k I_k(\widehat{f})|x|^{\left(\frac{\alpha-1}{\alpha}\right)k} = C_k|x|^{\left(\frac{\alpha-1}{\alpha}\right)k}$$
$$|\widehat{f}(\xi)| \leq (j+1)!(2^{2-\beta}\pi)^{j+1}\beta^j I_j(f)|\xi|^{\left(\frac{\beta-1}{\beta}\right)j} = \widehat{C}_j|\xi|^{\left(\frac{\beta-1}{\beta}\right)j},$$

where

$$I_k(g) = \int_{\mathbb{R}} |g(y)||y|^k \,\mathrm{d}y.$$

To get these bounds, we use that whenever a point x lies in an interval $[a_{m+1}^{(k)}, a_m^{(k)}]$ of two consecutive zeros of $f^(k)$, Fourier inversion implies

$$\begin{aligned}
|f^{(k)}(x)| &= |f^{(k)}(x) - f(a_m^{(k)})| \\
&= \left|\int_{\mathbb{R}} \widehat{f}(y)(2\pi iy)^k[e^{-2\pi iyx} - e^{-2\pi iya_m^{(k)}}]\,\mathrm{dy}\right| \\
&\leq (2\pi)^{k+1} I_{k+1}(\widehat{f})|x - a_m^{(k)}| \\
&\leq (2\pi)^{k+1} I_{k+1}(\widehat{f})|a_{m+1}^{(k)} - a_m^{(k)}|.
\end{aligned}$$

Ideas (Cont.)

(ii) Get bounds for I_k:

$$I_k(f) \leq G(k) + H(k) I_{k^*}(f),$$

where $k^* \sim \gamma k$, where $\gamma = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\beta}{1-\beta}\right)$ and G and H are well behaved.

(iii) Examine the conditions on α and β and obtain decay of the form

$$|f(x)| \leq C_f e^{-(1-\theta)|x|^A}.$$

where $A = A(\alpha, \beta)$.

(iii) This will imply $\widehat{f}$ can be extended to the whole complex plane as an analytic function with order at most $\frac{A}{A-1}$. That is, for all $\varepsilon > 0$,

$$|\widehat{f}(z)| \lesssim_\varepsilon e^{|z|^{\frac{A}{A-1}+\varepsilon}}.$$

Now one has to examine A and use Hadamard's theorem to conclude $\widehat{f} = 0$.

Questions

Maximal perturbations conjecture

Let $f \in \mathcal{S}_{even}(\mathbb{R})$ be a real function. Then there is $\theta > 0$ so that, if $|\varepsilon_i| < \theta$, $\forall\, i \in \mathbb{N}$, f can be uniquely recovered from its values

$$f(0), f(\sqrt{1+\varepsilon_1}), f(\sqrt{2+\varepsilon_2}), \dots,$$

together with the values of its Fourier transform

$$\widehat{f}(0), \widehat{f}(\sqrt{1+\varepsilon_1}), \widehat{f}(\sqrt{2+\varepsilon_2}), \dots.$$

Maximal perturbations conjecture - Weak form

Let $f \in \mathcal{S}_{even}(\mathbb{R})$ be a real function. Then, for each $a > 0$, there is $\delta > 0$ so that, if $|\varepsilon_i| \leq \delta k^{-a}$, then f can be uniquely recovered from its values

$$f(0), f(\sqrt{1+\varepsilon_1}), f(\sqrt{2+\varepsilon_2}), \dots,$$

together with the values of its Fourier transform

$$\widehat{f}(0), \widehat{f}(\sqrt{1+\varepsilon_1}), \widehat{f}(\sqrt{2+\varepsilon_2}), \dots.$$

Thank you!