Uniform Distribution on the Sphere and the Isotropic Discrepancy of Lattice Point Sets

based on joint work with F. Pillichshammer

Mathias Sonnleitner

2020-07-31

Overview

- Low-discrepancy sets on the sphere
- Mapping lattice point sets to the sphere
- Isotropic discrepancy and the spectral test

Discrepancy on the sphere

Let $\mathbb{S}^2$ be the unit sphere of $\mathbb{R}^3$. A (spherical) cap is a set

$$C = C(x,t) = \{z \in \mathbb{S}^2 : x \cdot z \geq t\}, \quad x \in \mathbb{S}^2, t \in (-1,1).$$

If $P \subset \mathbb{S}^2$ is a finite set of points, the discrepancy with respect to a cap C is denoted by

$$D(P,C) := \left| \frac{\#(P \cap C)}{\#P} - \sigma(C) \right|.$$

Here, σ is the normalized surface area measure. The spherical cap discrepancy of P is

$$D(P) := \sup_{C \text{ is a cap}} D(P,C).$$

Discrepancy on the sphere

The minimal spherical cap discrepancy is

$$D(N) := \inf_{\#P=N} D(P), \quad N \in \mathbb{N}.$$

Theorem (Beck, 1984)

There are constants $0 < c_1 < c_2 < \infty$ *such that*

$$c_1 \, N^{-3/4} \leq D(N) \leq c_2 \, N^{-3/4} \sqrt{\log N}, \quad N \in \mathbb{N}.$$

Open Question

What is the asymptotic behaviour of $(D(N))_{N\in\mathbb{N}}$ *as* $N \to \infty$*?*

The quest for low-discrepancy point sets

A sequence $P_N \subset \mathbb{S}^2, N \in \mathbb{N}$, of N-point sets is said to be of low-discrepancy if, for some $C > 0$,

$$D(P_N) \leq C\, N^{-3/4}\sqrt{\log N}, \quad N \in \mathbb{N}.$$

Goal

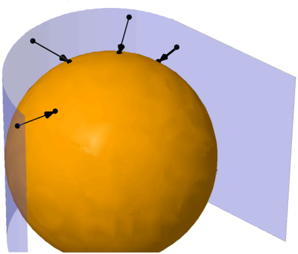
Construct a low-discrepancy sequence.

An incomplete list of results towards this goal:

- Lubotzky, Phillips, and Sarnak (1986): $D(P_N) \lesssim N^{-1/3}(\log N)^{2/3}$
- Aistleitner, Brauchart, and Dick (2012): $D(P_N) \lesssim N^{-1/2}$
- Etayo (2019): $D(P_N) \asymp N^{-1/2}$.

Mapping good planar points onto the sphere

Aistleitner, Brauchart, and Dick considered the Lambert cylindrical equal-area projection $\Phi : [0,1)^2 \to \mathbb{S}^2$ to map low-discrepancy sequences on $[0,1)^2$, the Fibonacci lattice point sets and certain digital nets, onto the sphere.



For any finite $P \subset [0,1)^2$:

$$D(\Phi(P)) \leq 11 J(P),$$

where $J(P)$ is the isotropic discrepancy of P.

The isotropic discrepancy

If $P \subset [0,1)^2$ is finite and $K \subset [0,1)^2$ convex,

$$D(P,K) := \left| \frac{\#(P \cap K)}{\#P} - \mathrm{vol}_2(K) \right|.$$

The isotropic discrepancy of P is

$$J(P) = \sup_{K \subset [0,1)^2 \text{ is convex}} D(P,K).$$

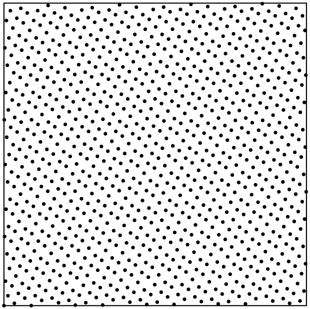
Theorem (Schmidt, 1975 and Beck, 1988)

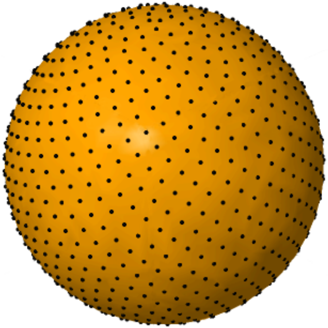
There are constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 N^{-2/3} \leq \inf_{\#P=N} J(P) \leq c_2 N^{-2/3} \log^4 N, \quad N \in \mathbb{N}.$$

The mapped Fibonacci lattice point set

Aistleitner, Brauchart, and Dick: Fibonacci lattice point set and certain digital nets satisfy $J(P_N) \lesssim N^{-1/2}$ and therefore $D(\Phi(P_N)) \lesssim N^{-1/2}$.





(Open) Question

Can we improve the bounds on $J(P_N)$ for the Fibonacci lattice point set and thus construct mapped point sets $\Phi(P_N)$ with cap discrepancy $D(\Phi(P_N))$ better than $N^{-1/2}$?

Our negative result

(Open) Question

Can we improve the bounds on $J(P_N)$ for the Fibonacci lattice point set and thus construct mapped point sets $\Phi(P_N)$ with cap discrepancy $D(\Phi(P_N))$ better than $N^{-1/2}$?

No, also not if you consider more general lattice point sets.

Theorem (Pillichshammer, S., 2019)

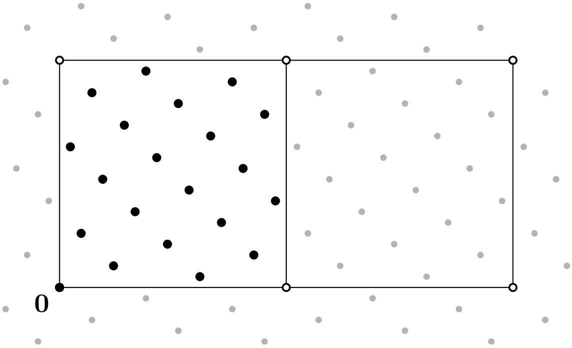
Let $\mathcal{P}(L)$ be an N-element lattice point set in $[0,1)^d$. Then we have

$$J(\mathcal{P}(L)) \geq c_d N^{-1/d} \quad \left(\gtrsim N^{-2/(d+1)} \right),$$

where $c_d := \frac{1}{2}\sqrt{\frac{\pi}{d}}\left(\Gamma\left(\frac{d}{2}+1\right)\right)^{-1/d}$.

Integration lattices and their lattice point sets

Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d$ be a basis of $\mathbb{R}^d$. An integration lattice is a lattice $L := \left\{ \sum_{i=1}^{d} k_i \boldsymbol{b}_i : k_i \in \mathbb{Z} \text{ for } i = 1, \ldots, d \right\}$ containing $\mathbb{Z}^d$.
The finite point set $\mathcal{P}(L) := L \cap [0,1)^d$ is the associated lattice point set.



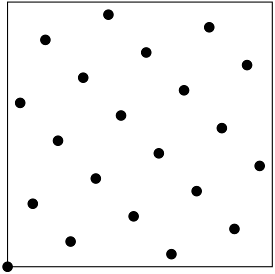
Dual lattice and spectral test

The dual lattice of an integration lattice L is given by

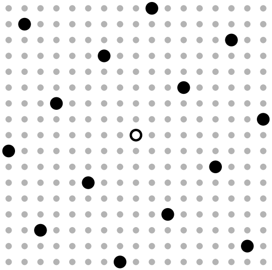
$$L^{\perp} := \{z \in \mathbb{Z}^d : z \cdot x \in \mathbb{Z} \text{ for all } x \in L\},$$

and the spectral test of L is defined by

$$\sigma(L) := \Big(\min_{z \in L^{\perp} \setminus \{\mathbf{0}\}} \|z\|_2 \Big)^{-1}.$$



$\mathcal{P}(L) \subset [0,1)^d$



$L^{\perp} \subset \mathbb{Z}^d$

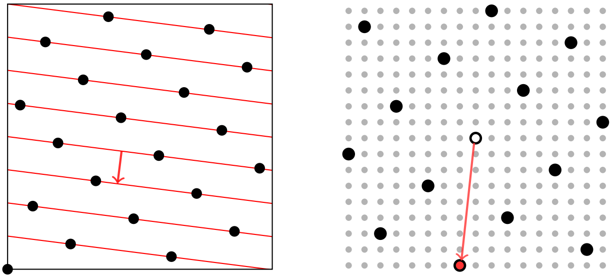
Dual lattice and spectral test

The dual lattice of an integration lattice L is given by

$$L^{\perp} := \{z \in \mathbb{Z}^d : z \cdot x \in \mathbb{Z} \text{ for all } x \in L\},$$

and the spectral test of L is defined by

$$\sigma(L) := \left(\min_{z \in L^{\perp} \setminus \{\mathbf{0}\}} \|z\|_2 \right)^{-1}.$$



$\mathcal{P}(L) \subset [0,1)^d$ $\qquad$ $L^{\perp} \subset \mathbb{Z}^d$

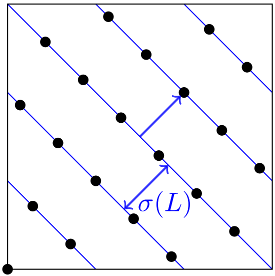
Dual lattice and spectral test

The dual lattice of an integration lattice L is given by

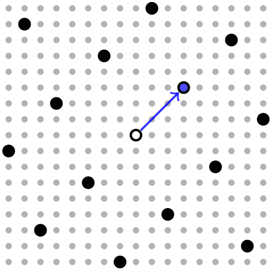
$$L^{\perp} := \{z \in \mathbb{Z}^d : z \cdot x \in \mathbb{Z} \text{ for all } x \in L\},$$

and the spectral test of L is defined by

$$\sigma(L) := \Big(\min_{z \in L^{\perp} \setminus \{\mathbf{0}\}} \|z\|_2 \Big)^{-1}.$$



$\mathcal{P}(L) \subset [0,1)^d$



$L^{\perp} \subset \mathbb{Z}^d$

A characterization based on the spectral test

Theorem (Pillichshammer, S., 2019)

Let $\mathcal{P}(L)$ be a lattice point set in $[0,1)^d$. Then we have

$$\frac{1}{\sqrt{d}}\,\sigma(L) \leq J(\mathcal{P}(L)) \leq d\,2^{2d+2}\sigma(L).$$

If $\sigma(L) \leq 1/2$, one can replace $1/\sqrt{d}$ by a constant.

Proposition

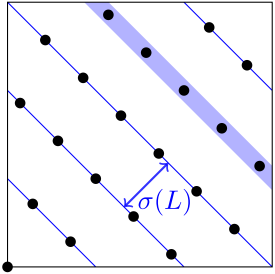
Let $\mathcal{P}(L)$ be an N-element lattice point set in $[0,1)^d$. Then we have

$$\sigma(L) \geq \frac{\sqrt{\pi}}{2}\left(\Gamma\left(\frac{d}{2}+1\right)\right)^{-1/d} N^{-1/d}.$$

The unconditional lower bound

At most $\lfloor \sqrt{d}/\sigma(L) \rfloor$ hyperplanes of a suitable cover intersect the unit cube $[0,1)^d$. By the pidgeonhole principle there exists a convex such that

$$D(\mathcal{P}(L), \;) = \frac{\#(\mathcal{P}(L) \cap \;)}{\#\mathcal{P}(L)} \geq \sigma(L)/\sqrt{d}.$$

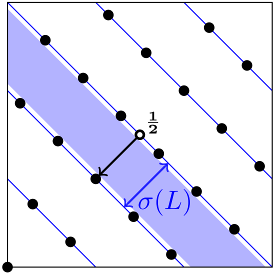


The improvement under $\sigma(L) \leq 1/2$

Let $\mathbf{\frac{1}{2}} := (\frac{1}{2}, \ldots, \frac{1}{2})$. We construct a convex with

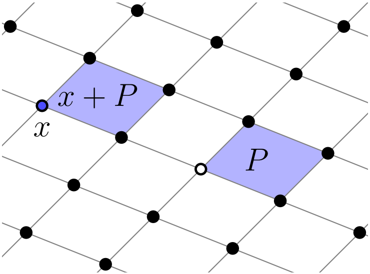
$$D(\mathcal{P}(L), \;) = \mathrm{vol}_d(\;)$$
$$\geq \inf_H \mathrm{vol}_{d-1}(H \cap [0,1)^d) \times \sigma(L) \geq \frac{1}{17}\,\sigma(L),$$

where H can be any hyperplane with $\mathrm{distance}(H, \mathbf{\frac{1}{2}}) \leq 1/2$.



Tessellations by unit cells

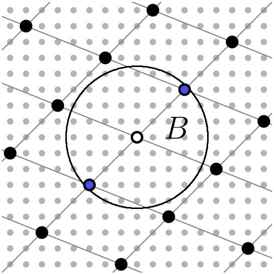
Given a lattice $L \subset \mathbb{R}^d$ with basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d$ the fundamental parallelotope is $P := \{\sum_{i=1}^{d} \lambda_i \boldsymbol{b}_i : 0 \leq \lambda_i < 1\}$.



Independent of the basis, $\mathrm{vol}_d(P) = \det(L)$ and for an integration lattice such that $\#\mathcal{P} = N$ we have $\det(L) = \frac{1}{N}$.

The lower bound of $\sigma(L)$

For any N-element lattice point set $\mathcal{P}(L)$: $\sigma(L) \geq c_d N^{-1/d}$.



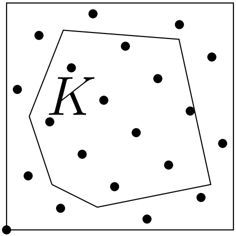
Minkowski: If $\mathrm{vol}_d(B) > 2^d \det(L^{\perp})$, then B contains at least one $z \in L^{\perp} \backslash \{\mathbf{0}\}$.

$$L \text{ integration lattice} \Rightarrow \det(L^{\perp}) = \det(L)^{-1} = N$$

implies that $\|\bullet\|_2 \leq C_d N^{1/d}$ and thus $\sigma(L) \geq C_d^{-1} N^{-1/d}$. $\square$

The upper bound

Let $K \subset [0,1)^d$ be convex and $\mathcal{P}(L) \subset [0,1)^d$ be an N-element lattice point set.

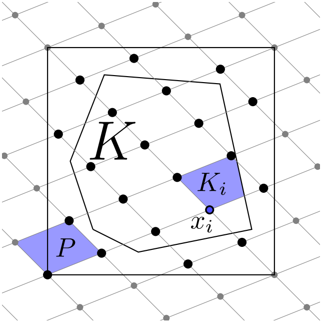


The upper bound

Let $K \subset [0,1)^d$ be convex and $\mathcal{P}(L) \subset [0,1)^d$ be an N-element lattice point set. We have

$$D(\mathcal{P}(L), K) = \sum_{i=1}^{M} D(\mathcal{P}(L), K_i),$$

where K_i is the intersection of C with a translated fundamental parallelotope $x_i + P$, $x_i \in L$.



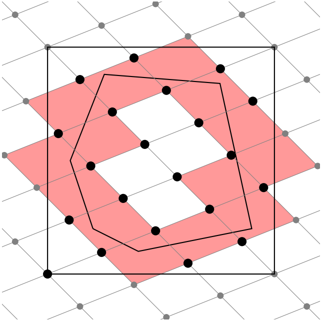
The upper bound

Then

$$D(\mathcal{P}(L), K) = \sum_{i=1}^{M} D(\mathcal{P}(L), K_i) \leq M_{\text{intersect}} \times \frac{1}{N},$$

where

$$M_{\text{intersect}} := \#\{i : K_i \cap \partial K \neq \emptyset\}.$$



The upper bound

Recall that

$$D(\mathcal{P}(L), K) \leq M_{\text{intersect}} \times \frac{1}{N}.$$

By a volumetric argument it holds that

$$M_{\text{intersect}} \leq \frac{\mathrm{vol}_d(\mathrm{diam}(P)\text{-neighbourhood of } \partial K)}{\mathrm{vol}_d(P) = \frac{1}{N}}.$$

After some convex geometry,

$$\mathrm{vol}_d(\mathrm{diam}(P)\text{-neighbourhood of } \partial K) \leq 2^{d+3}\mathrm{diam}(P).$$

Therefore,

$$D(\mathcal{P}(L), K) \leq 2^{d+3}\mathrm{diam}(P).$$

The upper bound

Recall that

$$D(\mathcal{P}(L), K) \leq 2^{d+3}\mathrm{diam}(P).$$

Using the LLL-algorithm, we have for P belonging to B

$$\mathrm{diam}(P) \leq d2^{d-1}\sigma(L),$$

where $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_d\}$ is a reduced basis of L. This is due to

$$\mathrm{diam}(P) \leq \sum_{i=1}^{d} \|\mathbf{b}_i\|_2 \leq d \max_{i=1,\ldots,d} \|\mathbf{b_i}\|_2 \leq d2^{d-1}\|\mathbf{b}_d^*\|_2,$$

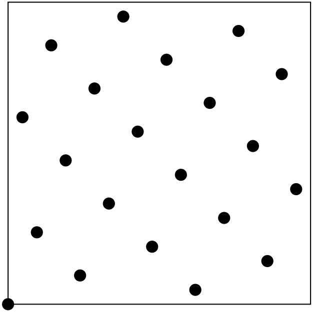
where $\mathbf{b}_d^*$ is the last vector in the Gram-Schmidt orthogonalization of B and thus related to a covering by hyperplanes. $\square$

Bonus: Covering Radius $\asymp$ Spectral Test

Proposition

For any lattice point set $\mathcal{P}(L) \subset [0,1)^d$ *we have for some* $C_d > 0$

$$\frac{1}{2}\sigma(L) \leq \sup_{y \in [0,1)^d} \min_{x \in \mathcal{P}(L)} \|x - y\|_2 \leq C_d \sigma(L).$$

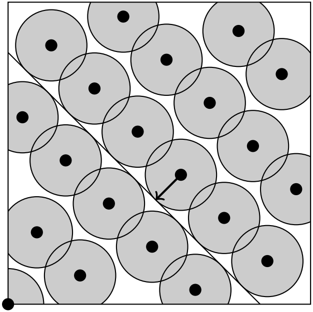


Bonus: Covering Radius $\asymp$ Spectral Test

Proposition

For any lattice point set $\mathcal{P}(L) \subset [0,1)^d$ we have for some $C_d > 0$

$$\frac{1}{2}\sigma(L) \leq \sup_{y \in [0,1)^d} \min_{x \in \mathcal{P}(L)} \|x - y\|_2 \leq C_d \sigma(L).$$

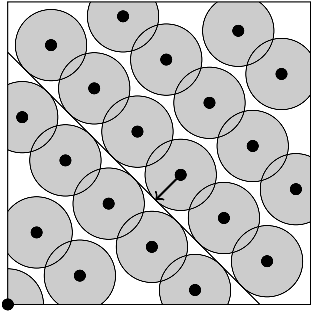


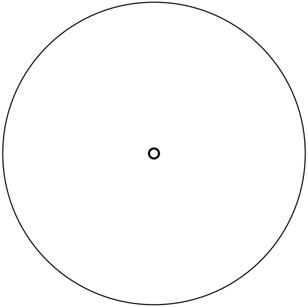
Bonus: Covering Radius $\asymp$ Spectral Test

Proposition

For any lattice point set $\mathcal{P}(L) \subset [0,1)^d$ *we have for some* $C_d > 0$

$$\frac{1}{2}\sigma(L) \leq \sup_{y\in[0,1)^d} \min_{x\in\mathcal{P}(L)} \|x-y\|_2 \leq C_d\sigma(L).$$



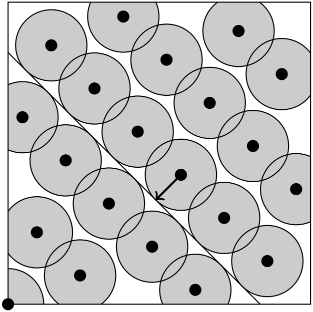


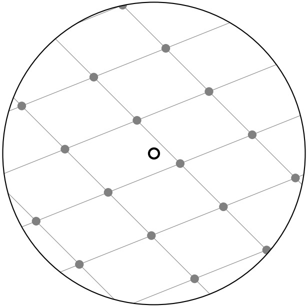
Bonus: Covering Radius $\asymp$ Spectral Test

Proposition

For any lattice point set $\mathcal{P}(L) \subset [0,1)^d$ we have for some $C_d > 0$

$$\frac{1}{2}\sigma(L) \leq \sup_{y \in [0,1)^d} \min_{x \in \mathcal{P}(L)} \|x - y\|_2 \leq C_d \sigma(L).$$



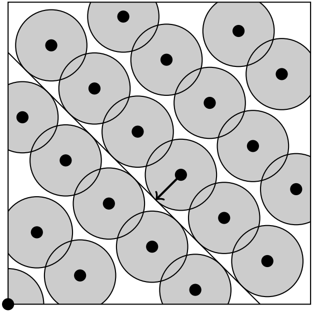


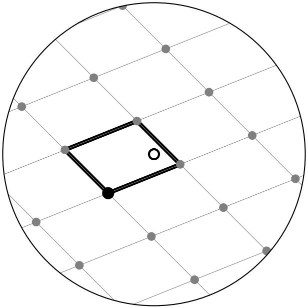
Bonus: Covering Radius $\asymp$ Spectral Test

Proposition

For any lattice point set $\mathcal{P}(L) \subset [0,1)^d$ we have for some $C_d > 0$

$$\frac{1}{2}\sigma(L) \leq \sup_{y\in[0,1)^d} \min_{x\in\mathcal{P}(L)} \|x-y\|_2 \leq C_d\sigma(L).$$





References I

C. Aistleitner, J. S. Brauchart, and J. Dick.
Point sets on the sphere $\mathbb{S}^2$ with small spherical cap discrepancy.
Discrete Comput. Geom., 48(4):990–1024, 2012.

K. Alishahi and M. Zamani.
The spherical ensemble and uniform distribution of points on the sphere.
Electron. J. Probab., 20(23), 2015.

J. Beck.
Some upper bounds in the theory of irregularities of distribution.
Acta Arith., 43(2):115–130, 1984.

References II

J. Beck.
Sums of distances between points on a sphere—an application of the theory of irregularities of distribution to discrete geometry.
Mathematika, 31(1):33–41, 1984.

J. Beck.
On the discrepancy of convex plane sets.
Monatsh. Math., 105(2):91–106, 1988.

U. Etayo.
Spherical cap discrepancy of the diamond ensemble.
arXiv-preprints, 2019.

A. Lubotzky, R. Phillips, and P. Sarnak.
Hecke operators and distributing points on S^2. II.
Comm. Pure Appl. Math., 40(4):401–420, 1987.

References III

F. Pillichshammer and M. Sonnleitner.
A note on isotropic discrepancy and spectral test of lattice point sets.
J. Complexity, 58:101441, 7, 2020.

W. M. Schmidt.
Irregularities of distribution. IX.
Acta Arith., 27:385–396, 1975.