Austin Anderson

A - compact , $\omega_N = \{x_1, \dots, x_N\} \subset A$

$$E_s(\omega_N) = \sum_{\substack{i,j=1\dots N \\ i \neq j}} \frac{1}{|x_i - x_j|^s} \longleftarrow$$

$$\mathcal{E}_s(A, N) = \min_{\omega_N \subset A} E_s(\omega_N)$$

$s < \dim(A)$

$\mathcal{E}_s(A, N) \sim N^2 \longleftarrow$ very general

$s > \dim(A) = d$

$\mathcal{E}_s(A, N) \asymp N^{1+s/d}$ $\nearrow$ $d \in \mathbb{N}$

$\frac{1}{|x|^s}$

If A is $\boxed{\text{smooth enough,}}$ then

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} = \frac{C_{s,d}}{H_d(A)^{s/d}}$$

$s = d$

$$\frac{\mathcal{E}_s(A,N)}{N^2 \log(N)}$$

What happens if $d \notin \mathbb{N}$?

History: 1) Steven Lalley

$$\delta(\omega_N) = \min_{\substack{i,j=1\dots N \\ i \neq j}} |x_i - x_j|$$

$$\delta(A, N) = \max_{\omega_N \subset A} \delta(\omega_N)$$

$$\lim_{s\to\infty} \left(\overset{\underline{\lim}/\overline{\lim}}{\lim_{N\to\infty}} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \right)^{1/s} = \frac{1}{\underline{\lim}/\overline{\lim}\ \lim_{N\to\infty} \delta(A,N)\, N^{1/d}}$$

(Borodachov, Saff)

For pretty general fractals,

$\lim \delta(A,N)\cdot N^{1/d}$ sometimes exists, $\leftarrow$

but sometimes exists only along $\leftarrow$
specific subsequences.

$\psi_1, \dots, \psi_n$ - contractions in $\mathbb{R}^d$

$r_1, \dots, r_n$ - contraction ratios

$\psi_j([0,1]^d)$ are disjoint.

There is a fractal (a.k.a. self-similar set A)
defined by $\psi_1, \dots, \psi_n$.

$1/3$ Cantor set: $\boxed{\psi_1(x) = \frac{x}{3}, \quad \psi_2(x) = \frac{x}{3} + \frac{2}{3}}$

If $\{t_1 \cdot \log(r_1) + \dots + t_n \log(r_n) \ : \ t_1, \dots, t_n \in \mathbb{Z}\}$
is dense in $\mathbb{R}$, then $\lim\limits_{N\to\infty} \delta(A,N)\cdot N^{1/d}$ exists

If $\{t_1 \cdot \log(r_1) + \dots + t_n \log(r_n)\} = h \cdot \mathbb{Z}$, then

$$\lim_{N\to\infty} \delta(A,N)\cdot N^{1/d} \text{ DNE}$$

(Lally's proof suggests $\nearrow$, because the
tools he uses say that this lim
exists only along some subsequences)

Borodachov-Saff : $r_1 = r_2 = r_3 = \dots = r_n$ - limit DNE

Anderson - A.R. : r_j's are "dependent" $\Rightarrow$ limit DNE

2) Borodachov – Saff

$r_1 = r_2 = \dots = r_n \implies$ limit DNE

3) Borodachov

A - fractal set, assume $\boxed{\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}}$ 

Then the optimal ω_N's will "converge" to the Hausdorff measure on A.

If ω_N's are optimal for $\mathcal{E}_s(A,N)$ along some subsequence of N's, and this subsequence attains the $\liminf\limits_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}$ then these ω_N's "converge" to the Hausdorff measure.

4) Vlasiuk – A.R. : $r_1 = r_2 = \dots = r_n = \mathcal{N}$

$\lim\limits_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\mathcal{E}(A,N)}{N^{1+s/d}}$ exists along some natural subsequences

$$N = \ell \boxed{r^{-d}}^K, \quad K = 1, \dots, \quad \ell \text{ is fixed},$$

We also explicitly show that, for example, when $r_1 = r_2 = 1/3$, then these limits are not equal for $\ell = 1$ and $\ell = 3$.

5) Anderson - A. R.

$r_1 = r^{i_1}, \dots, r_n = r^{i_n}, \quad \gcd(i_1, \dots, i_n) = 1$

$\log(r_j) = \boxed{i_j \cdot \log(r)}$

$\{t_1 \log(r_1) + \dots + t_n \log(r_n)\} = \underline{\log(r) \cdot \mathbb{Z}}$

For $N_k = \ell \cdot r^{-d \cdot k}$,

$$\lim_{k \to \infty} \frac{\mathcal{E}_s(A, N_k)}{N_k^{1+s/d}} \quad \text{exists}$$

Main question: if $\{t_1 \log(r_1) + \dots + t_n \log(r_n)\}$ is dense $\overset{?}{\Longrightarrow} \lim_{N \to \infty} \dfrac{\mathcal{E}_s(A, N)}{N^{1+s/d}}$ exists.

Main tool: renewal theorem

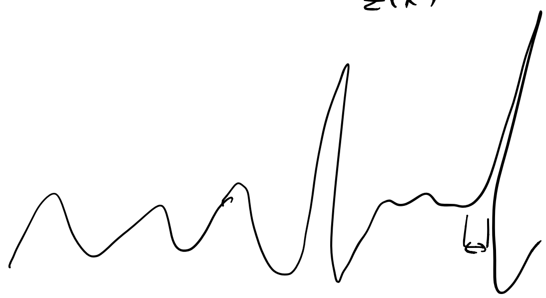
$Z(x)$, μ is a discrete probability measure with charges at $a_1, \dots, a_n$

$$\underline{Z(x)} - \underline{\int_0^x Z(x-t)\, d\mu(t)} = \boxed{z(x)}$$

Case 1: $a_1, \dots, a_n$ are independent

Then $\lim_{x\to\infty} Z(x)$ exists if

$z(x)$ is $\underbrace{\text{super- integrable}}$ from 0 to ∞



this does not happen

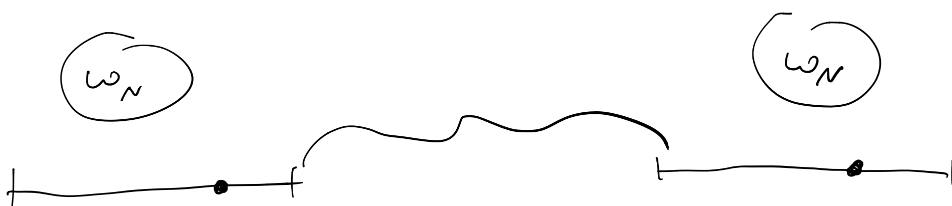
Case 2: $a_1, \dots, a_n$ are dependent

$$Z(n) - \sum_{k=1}^{n} \mu(k)\, Z(n-k) = z(n)$$

$\lim Z(n)$ exists if $\sum_{n=1}^{\infty} |z(n)| < \infty$

$$Z(k) = N_k^{-1-s/d} \cdot \mathcal{E}_s(A, N_k), \quad N_k = r^{-d\cdot k}$$

μ has weigts $\underbrace{r^{d\cdot i_k}}$ at $k = 1, \dots, n$



r^{-d}

$$\mathcal{E}_s(A, 2^{k+1}) \le \mathcal{E}_s(``\omega_N" \cup ``\omega_N") \le 2\, r^{-s} \cdot \mathcal{E}_s(A, 2^k)$$

+ error

$r^{-d-s} \cdot \mathcal{E}_s(A, 2^k)$ + error