Developments on the Fourier sign uncertainty principle

Oscar E. Quesada-Herrera (IMPA)

Point Distributions Webinar
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joint work with Emanuel Carneiro

PART I

Introduction

Fourier uncertainty

- Let $f \in L^1(\mathbb{R}^d)$. We define

$$\mathcal{F}_d[f](\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x)\,\mathrm{d}x \qquad (\xi \in \mathbb{R}^d).$$

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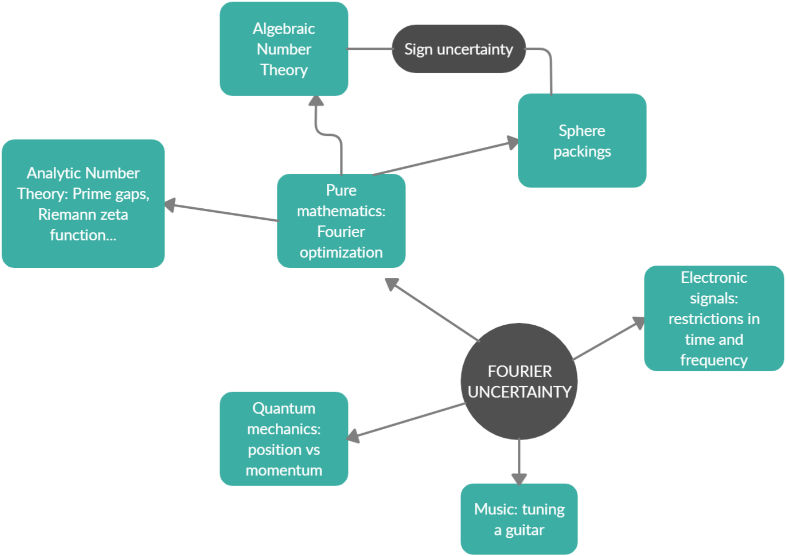

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- Heisenberg: the mass of f and $\widehat{f}$ cannot be arbitrarily concentrated near the origin

$$||f||_2^2 \leq \frac{4\pi}{d}|||x|f||_2 \cdot |||y|\widehat{f}||_2$$

Sign uncertainty principle

- Past examples: control the concentration of mass of f and $\widehat{f}$

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Question: Can we simultaneously control the signs of f and $\widehat{f}$?

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  - **Uncertainty:** What is the limit?
- **STEP 3**: Construct functions with those constraints that do the best possible job

Sphere Packings: Cohn-Elkies bounds

Theorem (Cohn, Elkies)

Let f be even, real-valued, and integrable on $\mathbb{R}^d$, such that $\widehat{f}$ is also integrable. Suppose that $f(0) = \widehat{f}(0) = 1$, $\widehat{f} \geq 0$ everywhere, and $f(x) \geq 0$ for $|x| \geq r$. Then, the optimal density of sphere packings Δ_d satisfies

$$\Delta_d \leq Vol(B_{\frac{r}{2}}),$$

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- **STEP 2**: Imposing the sign conditions on f and $\widehat{f}$, we recover desired info on minimal norm of lattice

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- **STEP 3**, sharp approach (Viazovska and Cohn, Kumar, Miller, Radchenko and Viazovska): Laplace transform of modular forms, satisfy hints of steps 1 and 2

Other examples of recipe

Analytic number theory:

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Algebraic number theory:

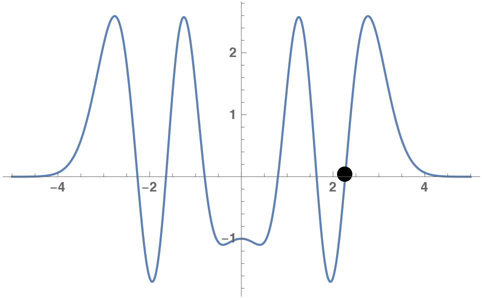
- **STEP 1**: Tate's zeta function is related to the Dedekind zeta function of a number field, an arbitrary f, and $\widehat{f}$

Sign uncertainty principle

Bourgain, Clozel and Kahane, 2010

- A continuous $f : \mathbb{R}^d \to \mathbb{R}$ is eventually non-negative if $f(x) \geq 0$ for sufficiently large $|x|$, and we define

$$r(f) := \inf\{r > 0 \ : \ f(x) \geq 0 \text{ for all } |x| \geq r\}.$$



$$f(x) = (x^{10} - 8x^8 + 15x^6 - x^4 - 2x^2 - 1)e^{-x^2}$$

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- Define

$$\mathbb{A}_{+1}(d) := \inf_{f \in \mathcal{A}_{+1}(d)} \sqrt{r(f)\, r\big(\widehat{f}\big)}.$$

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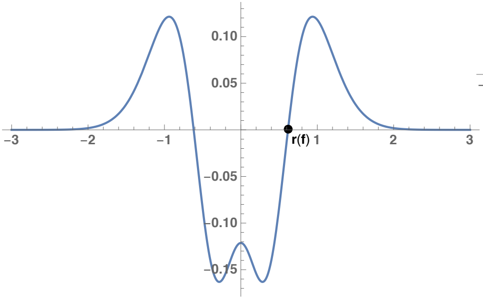
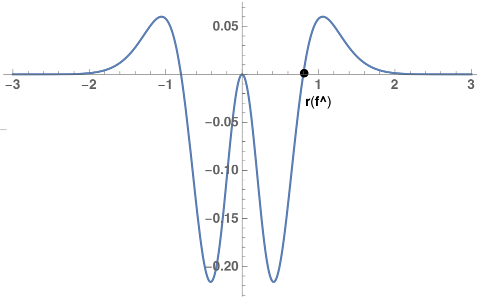
- They show:

$$\sqrt{\frac{d+2}{2\pi}} \geq \mathbb{A}_{+1}(d) \geq \sqrt{\frac{d}{2\pi e}}.$$

Pictures

Take for example $f(x) = e^{-\pi x^2/2} + e^{-2\pi x^2} - \left(\sqrt{2} + \frac{1}{\sqrt{2}}\right)e^{-\pi x^2}$.





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Gonçalves, Oliveira e Silva, Steinerberger 2017:

- Improved estimates
- Proved existence of radial extremizers
- Qualitative properties of extremizers

A dual sign uncertainty principle

Cohn and Gonçalves, 2019

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- They show:

$$C\sqrt{d} \ge \mathbb{A}_s(d) \ge c\sqrt{d}.$$

Sharp constants

Theorem

(i) *(Corollaries of Cohn and Elkies '03 ($d = 1$), Viazovska '17 ($d = 8$) and Cohn, Kumar, Miller, Radchenko and Viazovska '17 ($d = 24$))*

$$\mathbb{A}_{-1}(1) = 1\ ;\ \mathbb{A}_{-1}(8) = \sqrt{2}\ ;\ \mathbb{A}_{-1}(24) = 2.$$

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- Note that $\mathbb{A}_{+1}(12) = \mathbb{A}_{-1}(8)$. Conjecture (Cohn, Gonçalves):

$$\mathbb{A}_{+1}(d+4) \approx \mathbb{A}_{-1}(d).$$

- Gonçalves, Oliveira e Silva and Ramos (preprint, 2020); extensions of the (± 1)- sign uncertainty to a general operator setting.

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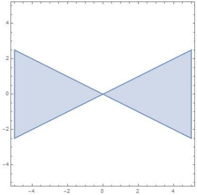
- More Fourier weighted norm inequalities, e.g. by Beckner, Benedetto, and others

PART II

Weighted sign Fourier uncertainty: the classical path

A "Certainty" principle: controlling the signs of f and $\widehat{f}$

- New control: can f and $\widehat{f}$ be positive in a given geometrical region, negative in its complement? If yes, how quickly?



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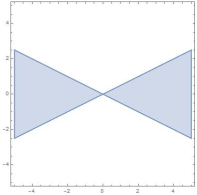

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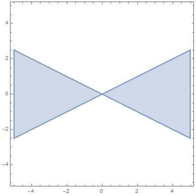

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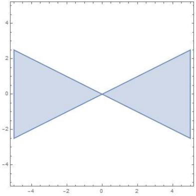

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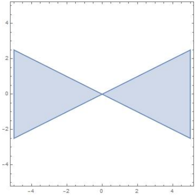

- **Partial answer:** If $A = 1$ **YES!**
- **Open question:** What happens if $A \neq 1$?

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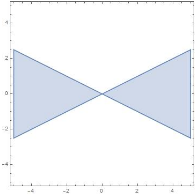

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A "Certainty" principle: controlling the signs of f and $\widehat{f}$

- New control: can f and $\widehat{f}$ be positive in a given geometrical region, negative in its complement? If yes, how quickly?
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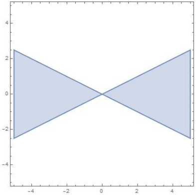

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- **Partial answer:** If P is harmonic, **yes!**

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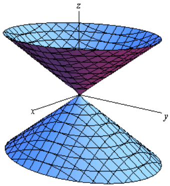

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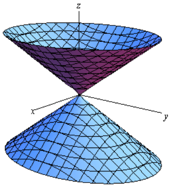

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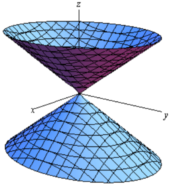

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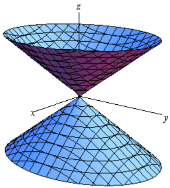

- More flexibility, but **What is Step 0???**

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- A measurable $g : \mathbb{R}^d \to \mathbb{R}$ is *eventually non-negative* if $g(x) \geq 0$ for sufficiently large $|x|$, and we define

$$r(g) := \inf\{r > 0 \ : \ g(x) \geq 0 \textit{ for all } |x| \geq r\}.$$

The function P

Let $P : \mathbb{R}^d \to \mathbb{R}$ be measurable function such that:

(P1) $P \in L^1_{\text{loc}}(\mathbb{R}^d)$.

(P2) P is either even or odd. We let $\mathfrak{r} \in \{0, 1\}$ be such that

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The generalized setup

Let $s \in \{+1, -1\}$. Consider the family:

$$\mathcal{A}_s(P;d) = \left\{ \begin{array}{l} f \in L^1(\mathbb{R}^d)\setminus\{0\} \text{ continuous, real-valued; } f(-x) = (-1)^{\mathfrak{r}} f(x); \\ \widehat{f}, Pf, P\widehat{f} \in L^1(\mathbb{R}^d); \\ \int_{\mathbb{R}^d} Pf \leq 0 \ , \ \int_{\mathbb{R}^d} s(-i)^{\mathfrak{r}} P\widehat{f} \leq 0; \\ Pf, \ s(-i)^{\mathfrak{r}} P\widehat{f} \text{ are eventually non-negative.} \end{array} \right\}.$$

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- Assuming technical but very general conditions, we may reduce to eigenfunctions, for all possible eigenvalues!! (P3)

Generalized eigenfunction problem

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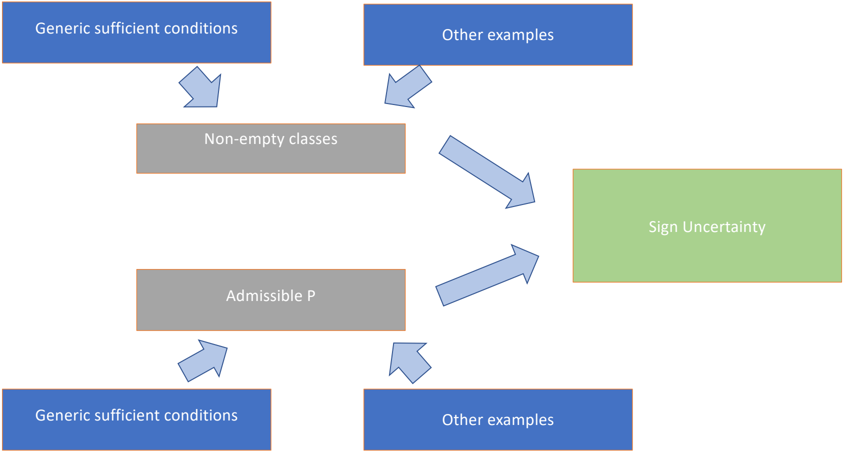
The classical path



Non-empty classes

Theorem (Non-empty classes)

Let P be such that $P\,e^{-\lambda\pi|\cdot|^2} \in L^1(\mathbb{R}^d)$ for all $\lambda > 0$. Assume that $P = H \cdot Q$, where $H : \mathbb{R}^d \to \mathbb{R}$ is a homogeneous and harmonic polynomial of degree $\ell \geq 0$, and $Q : \mathbb{R}^d \to \mathbb{R}$ is eventually non-negative. Then $\mathcal{A}_s^*(P; d)$ is non-empty.

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Theorem (Upper bounds)

Under the previous conditions, if P is also homogeneous of degree $\gamma > -d$, *we have the upper bounds*

$$\mathbb{A}_s^*(P; d) \leq \sqrt{\frac{\max\{d + \ell + \gamma\,,\, \ell - \gamma\}}{2\pi}} + O(1),$$

where the implied constant is universal. In fact, when $s\, i^{\ell+\mathfrak{r}} = -1$ *and* $-d < \gamma \leq -\frac{d}{2}$ *we have*

$$\mathbb{A}_s^*(P; d) = 0.$$

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Lemma (Bochner's relation)

Let $H : \mathbb{R}^d \to \mathbb{R}$ be a homogeneous, harmonic polynomial of degree ℓ, and $h : [0, \infty) \to \mathbb{R}$ be a function such that

$$\int_0^\infty |h(r)|^2 \, r^{d+2\ell-1} \mathrm{d}r < \infty.$$

Let $h_d : \mathbb{R}^d \to \mathbb{R}$ be the radial function on $\mathbb{R}^d$ induced by h, that is $h_d(x) := h(|x|)$. Then

$$\mathcal{F}_d[H \cdot h_d](\xi) = (-i)^\ell H(\xi) \cdot \mathcal{F}_{d+2\ell}[h_{d+2\ell}](\xi, 0),$$

where $\xi \in \mathbb{R}^d$ and $(\xi, 0) \in \mathbb{R}^d \times \mathbb{R}^{2\ell}$.

Proof: Non-empty classes and upper bounds.

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- Asymptotic analysis: optimize over several parameters to obtain upper bounds
- Note: different qualitative behaviour for $s = \pm 1$ when $\gamma \leq -\frac{d}{2}$

$\square$

More questions...

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- Then, necessarily, $f(0, y) = 0$ for all $y \in \mathbb{R}$?

Admissible functions

Definition (Admissible functions)

P is admissible if there there exists $1 \leq q \leq \infty$ and a positive constant $C = C(P; d; q)$ such that:

(i) *For all $f \in L^1(\mathbb{R}^d)$, with $\widehat{f} = \pm i^{\mathfrak{r}} f$ and $Pf \in L^1(\mathbb{R}^d)$, we have*

$$||f||_q \leq C \, \|Pf\|_1. \tag{1}$$

(ii) *If $q > 1$ then $P \in L^{q'}_{\text{loc}}(\mathbb{R}^d)$. If $q = 1$ we have $\lim_{r \to 0^+} \|P\|_{L^\infty(B_r)} = 0$.*

Theorem (Sufficient conditions for admissibility)

Let P be such that the sub-level set $A_\lambda = \{x \in \mathbb{R}^d \,:\, |P(x)| \leq \lambda\}$ has finite Lebesgue measure for some $\lambda > 0$. Then inequality (1) holds with $q = 1$. In particular, P is admissible with respect to $q = \infty$. If P is also homogeneous of degree $\gamma > 0$, we can take

$$C = (1 + \tfrac{\gamma}{d}) \left[\left(1 + \frac{d}{\gamma}\right) |A_1| \right]^{\frac{\gamma}{d}}.$$

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- In the non-homogeneous case, this inequality is replaced by the more general (but less explicit) Amrein-Berthier inequality: when E, F have finite measure,

$$\int_{\mathbb{R}^d} |g(x)|^2 \,\mathrm{d}x \leq C \left(\int_{E^c} |g(x)|^2 \,\mathrm{d}x + \int_{F^c} |\widehat{g}(x)|^2 \,\mathrm{d}x \right).$$

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Sign uncertainty

Theorem

Assume that the class $\mathcal{A}_s^*(P;d)$ is non-empty and that P is admissible with respect to an exponent $1 \leq q \leq \infty$. Then there exists a positive constant $C^* = C^*(P;d;q)$ such that

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- Also: general conditions that show the existence of extremizers

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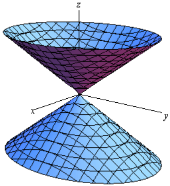

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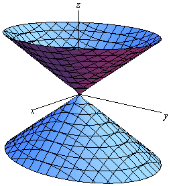

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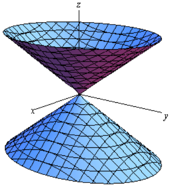

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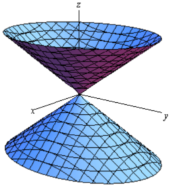

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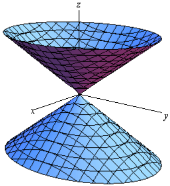

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- **Open question:** Can we get lower bound $c\sqrt{d+\gamma}$?

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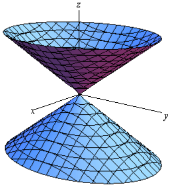
- $P(x, y, z) = (x^2 + y^2 - 2z^2)(x^2 + y^2 + 2z^2)$



- We get sign uncertainty for (non-empty) family of functions eventually positive inside the cone, eventually negative outside!
- Let $P(x) = |x|^\gamma$, with $\gamma \geq 0$.
- Combining the three theorems in homogeneous case, we get

$$C\sqrt{d+\gamma} \geq \mathbb{A}_s(|x|^\gamma, d) \geq c\sqrt{d}$$

- **Open question:** Can we get lower bound $c\sqrt{d+\gamma}$?
- Logarithmic weight: $P(x) = |x|^\gamma \log |x|$.

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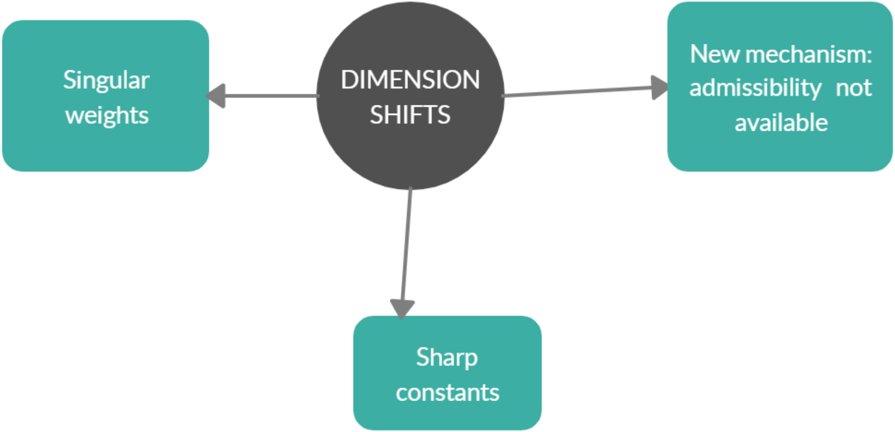
- There are counterexamples (communicated by Fedor Nazarov)
- Hard to establish uncertainty for $\gamma < 0$ with classical methods
- Dimension shifts: we can solve it for all negative $\gamma > -d$ outside small neighborhood of $-\frac{d}{2}$!

PART III

Dimension shifts

Dimension shifts



Theorem (Dimension shifts)

Let $\ell \geq 0$ and $\mathfrak{r}(\ell) \in \{0,1\}$ be such that $\mathfrak{r}(\ell) \equiv \ell \pmod 2$. Let $P : \mathbb{R}^{d+2\ell} \to \mathbb{R}$ be a radial function verifying (P3)*. Write $P(x) = P_0(|x|)$. Let $\widetilde{P} : \mathbb{R}^d \to \mathbb{R}$ be of the form*

$$\widetilde{P}(x) = H(x)\, P_0(|x|)\, Q(x),$$

where $H : \mathbb{R}^d \to \mathbb{R}$ is a non-zero homogeneous and harmonic polynomial of degree ℓ and $Q : \mathbb{R}^d \to \mathbb{R}$ is an even non-negative function, homogeneous of degree 0. If $\mathcal{A}_s^*(P; d+2\ell)$ is non-empty, then $\mathcal{A}^*_{s(-1)^{(\mathfrak{r}(\ell)+\ell)/2}}\big(\widetilde{P}; d\big)$ is also non-empty and

$$\mathbb{A}_s^*(P; d+2\ell) \geq \mathbb{A}^*_{s(-1)^{(\mathfrak{r}(\ell)+\ell)/2}}\big(\widetilde{P}; d\big).$$

If P has a bounded sub-level set, $Q \equiv 1$ and $H \in O(d)(x_1 x_2 \ldots x_\ell)$ $(0 \leq \ell \leq d)$, the equality holds.

Useful: $Q = |x|^\ell \operatorname{sgn}(H)/H$, when $P(x) = |x|^\gamma$, $\gamma < 0$.

Corollary (Sharp constants)

Let $\mathfrak{r}(\ell) \in \{0, 1\}$ *be such that* $\mathfrak{r}(\ell) \equiv \ell \pmod 2$. *Then*

$$\mathbb{A}_{(-1)^{(\mathfrak{r}(\ell)+\ell+2)/2}}(R(x_1 \ldots x_\ell)\,; 8 - 2\ell) = \sqrt{2}, \quad 0 \le \ell \le 2; \quad R \in O(8 - 2\ell);$$

$$\mathbb{A}_{(-1)^{(\mathfrak{r}(\ell)+\ell)/2}}(R(x_1 \ldots x_\ell)\,; 12 - 2\ell) = \sqrt{2}, \quad 0 \le \ell \le 4; \quad R \in O(12 - 2\ell);$$

$$\mathbb{A}_{(-1)^{(\mathfrak{r}(\ell)+\ell+2)/2}}(R(x_1 \ldots x_\ell)\,; 24 - 2\ell) = 2, \quad 0 \le \ell \le 8; \quad R \in O(24 - 2\ell);.$$

Open Problem: dimension shifts and Cohn-Gonçalves conjectures

Conjecture (Cohn, Gonçalves)

The following limits exist and are equal:

$$\lim_{d\to\infty} \frac{\mathbb{A}_{+1}(d)}{\sqrt{d}} = \lim_{d\to\infty} \frac{\mathbb{A}_{-1}(d)}{\sqrt{d}}$$

- Numerical evidence by Cohn and Gonçalves suggests even more: approximately,

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- Open question: Can we relate the problems with two different weights, in the same dimension, same eigenvalue?

THANK YOU!