Sampling recovery of functions from reproducing kernel Hilbert spaces in the uniform norm

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$f\colon D \subset \mathbb{R}^d \to \mathbb{C}$ $\qquad$ $f(\mathbf{X})$: $\ f(\mathbf{x}^1), \dots, f(\mathbf{x}^n)$

Reconstruct f from samples

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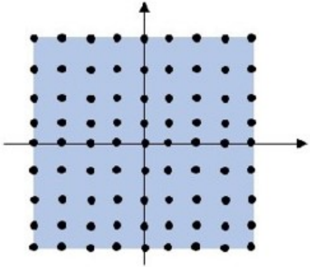
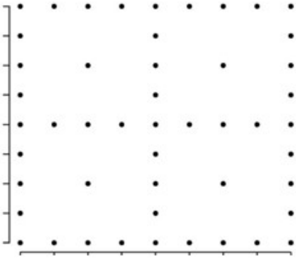
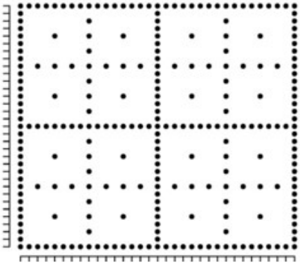
Reconstruct f from samples







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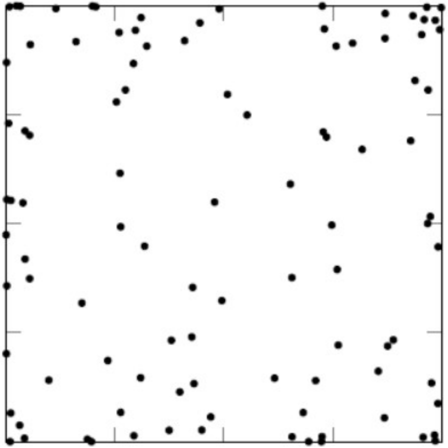

- The sampling nodes should work for a class of functions simultaneously
- $H(K)$ is a reproducing kernel Hilbert space (RKHS)

- Control the worst-case error

$$\sup_{\|f\|_{H(K)} \leq 1} \|f - S^m_{\mathbf{X}} f\|_{\ell_\infty(D)}$$

- Discuss the power of standard information in the uniform norm
- Obtain new recovery guarantees for concrete Sobolev type spaces

Linear information: Gelfand numbers / widths, approximation numbers / linear widths

$$a_n\left(\mathrm{Id}\colon H(K)\to F\right) := \inf_{\substack{A\in\mathcal{L}(H(K),F)\\ \mathrm{rank}A<n}} \sup_{\|f\|_{H(K)}\le 1} \|f-Af\|_F \tag{1}$$

Standard information: sampling numbers

$$g_n\left(\mathrm{Id}\colon H(K)\to F\right) := \inf_{\mathbf{X}=\{\mathbf{x^1},\dots,\mathbf{x}^n\}} \inf_{R\in\mathcal{L}(\mathbb{C}^n,F)} \sup_{\|f\|_{H(K)}\le 1} \|f-R\left(f(\mathbf{X})\right)\|_F \tag{2}$$

$$a_n(\mathrm{Id}) \le g_n(\mathrm{Id})$$

$$\|f\|_{L_2(D,\varrho_D)} = \left(\int_D |f(\boldsymbol{x})|^2 \mathrm{d}\varrho_D(\boldsymbol{x})\right)^{1/2}$$

$$\|f\|_{\ell_\infty(D)} = \sup_{\boldsymbol{x}\in D} |f(\boldsymbol{x})|$$

$$\forall f \in H(K),\ \forall \boldsymbol{x} \in D \qquad f(\boldsymbol{x}) = (f, K(\cdot, \boldsymbol{x}))_{H(K)}$$

$$\|K\|_\infty^2 := \sup_{\boldsymbol{x}\in D} K(\boldsymbol{x}, \boldsymbol{x}) < \infty \tag{3}$$

$$\|f\|_{\ell_\infty(D)} \leq \|K\|_\infty \cdot \|f\|_{H(K)}$$

$$\operatorname{tr} K := \|K\|_2^2 = \int_D K(\boldsymbol{x}, \boldsymbol{x}) \mathrm{d}\varrho_D(\boldsymbol{x}) < \infty \tag{4}$$

$$\forall f \in H(K),\ \forall \boldsymbol{x} \in D \qquad f(\boldsymbol{x}) = (f, K(\cdot, \boldsymbol{x}))_{H(K)}$$

$$\mathrm{Id}\colon H(K) \to L_2(D, \varrho_D), \qquad W_{\varrho_D} = \mathrm{Id}^* \circ \mathrm{Id}\colon H(K) \to H(K),$$

where $(f, \mathrm{Id} g)_{L_2(D,\varrho_D)} = (\mathrm{Id}^* f, g)_{H(K)}$.

$(\lambda_n)_{n=1}^{\infty}$ — rearrangement of eigenvalues of W_{ϱ_D}, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$,
$(\sigma_n)_{n=1}^{\infty}$ — set of singular values, i.e., $\sigma_j = \sqrt{\lambda_j}$, $j = 1, 2, \ldots$,

$(e_n(\mathbf{x}))_{n=1}^{\infty} \subset H(K)$ — set of right singular functions,
$(\eta_n(\mathbf{x}))_{n=1}^{\infty} = (\sigma_n^{-1} e_n(\mathbf{x}))_{n=1}^{\infty} \subset L_2(D, \varrho_D)$.

Mercer kernel

$$K(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{\infty} \overline{e_k(\mathbf{y})} e_k(\mathbf{x}), \qquad \mathbf{x}, \mathbf{y} \in D \tag{5}$$

Least squares algorithm

D. Krieg, M. Ullrich Function values are enough for L_2-approximation, *Found. Comp. Math.*, 21: 1141–1151, 2021.

L. Kämmerer, T. Ullrich, T. Volkmer Worst case recovery guarantees for least squares approximation using random samples, *Constr. Approx.*, 2021.

Recovery operator $S_{\mathbf{X}}^m := \sum_{k=1}^{m-1} c_k \eta_k$

$\mathbf{f} := (f(\mathbf{x}^1), \ldots, f(\mathbf{x}^n))^\top$, $\mathbf{c} := (c_1, \ldots, c_{m-1})^\top$, $(\eta_k(\mathbf{x}))_{k=1}^\infty = (\sigma_k^{-1} e_k(\mathbf{x}))_{k=1}^\infty$

$$\mathbf{L}_{n,m} := \mathbf{L}_{n,m}(\mathbf{X}) = \begin{pmatrix} \eta_1(\mathbf{x}^1) & \eta_2(\mathbf{x}^1) & \cdots & \eta_{m-1}(\mathbf{x}^1) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1(\mathbf{x}^n) & \eta_2(\mathbf{x}^n) & \cdots & \eta_{m-1}(\mathbf{x}^n) \end{pmatrix} \tag{6}$$

Solve the over-determined linear system

$$\mathbf{L}_{n,m} \cdot \mathbf{c} = \mathbf{f}$$

via least squares, i.e., compute

$$\mathbf{c} = (\mathbf{L}_{n,m}^* \mathbf{L}_{n,m})^{-1} \mathbf{L}_{n,m}^* \cdot \mathbf{f} \tag{7}$$

Weighted least squares regression

Input: $\mathbf{X} = (\mathbf{x}^1, ..., \mathbf{x}^n) \in D^n$ set of distinct sampling nodes,
$\mathrm{f} = (f(\mathbf{x}^1), ..., f(\mathbf{x}^n))^\top$ samples of f evaluated at the nodes from $\mathbf{X}$,
$m \in \mathbb{N}$ $m < n$ such that the matrix $\widetilde{\mathbf{L}}_{n,m}$ has full (column) rank.

Compute reweighted samples $\boldsymbol{g} := (g_j)_{j=1}^n$ with

$$g_j := \begin{cases} 0, & \varrho_m(\mathbf{x}^j) = 0, \\ f(\mathbf{x}^j)/\sqrt{\varrho_m(\mathbf{x}^j)}, & \varrho_m(\mathbf{x}^j) \neq 0\,. \end{cases}$$

Solve the over-determined linear system $\widetilde{\mathbf{L}}_{n,m} \cdot (\tilde{c}_1, ..., \tilde{c}_{m-1})^\top = \mathrm{g}\,,$

$$\widetilde{\mathbf{L}}_{n,m} := \left(l_{j,k}\right)_{j=1,k=1}^{n,m-1}, \quad l_{j,k} := \begin{cases} 0, & \varrho_m(\mathbf{x}^j) = 0, \\ \eta_k(\mathbf{x}^j)/\sqrt{\varrho_m(\mathbf{x}^j)}, & \varrho_m(\mathbf{x}^j) \neq 0, \end{cases}$$

via least squares, i.e., compute $(\tilde{c}_1, ..., \tilde{c}_{m-1})^\top := (\widetilde{\mathbf{L}}^*_{n,m}\widetilde{\mathbf{L}}_{n,m})^{-1}\,\widetilde{\mathbf{L}}^*_{n,m} \cdot \mathbf{g}$.
Output: $\tilde{\mathbf{c}} = (\tilde{c}_1, ..., \tilde{c}_{m-1})^\top \in \mathbb{C}^{m-1}$ coefficients of the approximant

$$\widetilde{S}^m_{\mathbf{X}} f := \sum_{k=1}^{m-1} \tilde{c}_k \eta_k\,.$$

D. Krieg, M. Ullrich Function values are enough for L_2-approximation, *Found. Comp. Math.*, 21: 1141–1151, 2021.

A. Cohen, G. Migliorati Optimal weighted least-squares methods. *SMAI J. Comput. Math.*, 3: 181–203, 2017.

V. N. Temlyakov On optimal recovery in L_2. *J. Complexity*, 65, 2021.

$$\varrho_m(\mathrm{x}) = \frac{1}{2}\Big(\frac{1}{m-1}\sum_{k=1}^{m-1}|\eta_k(\mathrm{x})|^2 + \frac{1}{\sum_{k=m}^{\infty}\lambda_k}\sum_{k=m}^{\infty}|e_k(\mathrm{x})|^2\Big) \qquad (8)$$

$$\varrho'_m(\mathrm{x}) = \frac{1}{2(m-1)}\sum_{k=1}^{m-1}|\eta_k(\mathrm{x})|^2 + \frac{1}{2} \qquad (9)$$

$\mathbf{X} = (\boldsymbol{x}^1, ..., \boldsymbol{x}^n)$ are drawn i.i.d. with respect to $\varrho_m(\cdot)d\varrho_D$

For every $f \in H(K)$ with a Mercer kernel K, it holds

$$f(\mathbf{x}) = \sum_{k=1}^{\infty} (f, e_k)_{H(K)} e_k(\mathbf{x}).$$

Let

$$P_m f := \sum_{k=1}^{m} (f, e_k)_{H(K)} e_k(\cdot)$$

be the projection onto the space $\mathrm{span}\{e_1(\cdot), \ldots, e_m(\cdot)\}$.

$$\|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)} \leq \|f - P_{m-1} f\|_{\ell_\infty(D)} + \|P_{m-1} f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}$$

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$$\sup_{\|f\|_{H(K)} \le 1} \|f - P_{m-1} f\|_{\ell_\infty(D)} \le \sqrt{2 \sum_{k \ge \lfloor m/4 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}}, \qquad (10)$$

where

$$N_{K,\varrho_D}(m) := \sup_{\mathbf{x} \in D} N_{K,\varrho_D}(m, \mathbf{x}) = \sup_{\mathbf{x} \in D} \sum_{k=1}^{m-1} |\eta_k(\mathbf{x})|^2 .$$

$$\sup_{\|f\|_{H(K)}\le 1} \|f - P_{m-1} f\|^2_{\ell_\infty(D)} = \sup_{\mathbf{x}\in D} \sum_{k\ge m} |e_k(\mathbf{x})|^2$$

$$\le \sup_{\mathbf{x}\in D} \sum_{l=\lfloor \log_2 m\rfloor}^{\infty} \sum_{2^l\le k<2^{l+1}} |\sigma_k \eta_k(\mathbf{x})|^2$$

For all $2^l \le k < 2^{l+1}$ it holds $\sigma_k^2 \le \frac{1}{2^{l-1}} \sum_{2^{l-1}\le j<2^l} \sigma_j^2$

$$\le \sum_{l=\lfloor \log_2 m\rfloor}^{\infty} \frac{N_{K,\varrho_D}(2^{l+1})}{2^{l-1}} \sum_{2^{l-1}\le j<2^l} \sigma_j^2 \le \cdots \le 2 \sum_{k\ge\lfloor m/4\rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}$$

$$\|P_{m-1}f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)} = \|\widetilde{S}_{\mathbf{X}}^m (f - P_{m-1}f)\|_{\ell_\infty(D)} = \left\| \sum_{k=1}^{m-1} \tilde{c}_k \eta_k(\mathbf{x}) \right\|_{\ell_\infty(D)}$$

$$\leq \sqrt{N_{K,\varrho_D}(m) \cdot \sum_{k=1}^{m-1} |\tilde{c}_k|^2}$$

$$\|P_{m-1}f - \widetilde{S}^m_{\mathbf{X}} f\|_{\ell_\infty(D)} = \|\widetilde{S}^m_{\mathbf{X}}(f - P_{m-1}f)\|_{\ell_\infty(D)} = \left\| \sum_{k=1}^{m-1} \tilde{c}_k \eta_k(\mathbf{x}) \right\|_{\ell_\infty(D)}$$

$$\leq \sqrt{N_{K,\varrho_D}(m) \cdot \sum_{k=1}^{m-1} |\tilde{c}_k|^2}$$

$$(\tilde{c}_1, \ldots, \tilde{c}_{m-1})^\top = (\widetilde{\mathbf{L}}^*_{n,m} \widetilde{\mathbf{L}}_{n,m})^{-1} \widetilde{\mathbf{L}}^*_{n,m} \cdot \mathbf{g}, \quad \boldsymbol{g} = (g_j)_{j=1}^n \text{ with}$$

$$g_j := \begin{cases} 0, & \varrho_m(\mathbf{x}^j) = 0, \\ (f - P_{m-1}f)(\mathbf{x}^j)/\sqrt{\varrho_m(\mathbf{x}^j)}, & \varrho_m(\mathbf{x}^j) \neq 0\,. \end{cases} \tag{11}$$

N. Nagel, M. Schäfer, T. Ullrich A new upper bound for sampling numbers. *Found. Comp. Math.*, 2021.

$$N(m) \leq n/(10 r \log n), \; r > 1 \quad \Longrightarrow \quad \left\| (\mathbf{L}^*_{n,m} \mathbf{L}_{n,m})^{-1} \mathbf{L}^*_{n,m} \right\|_{2\to 2} \leq \sqrt{2/n}$$

$$\varrho_m(\mathbf{x}) = \frac{1}{2}\left(\frac{1}{m-1} \sum_{k=1}^{m-1} |\eta_k(\mathbf{x})|^2 + \frac{1}{\sum_{k=m}^\infty \lambda_k} \sum_{k=m}^\infty |e_k(\mathbf{x})|^2 \right)$$

$$\widetilde{N}(m) := \sup_{\mathbf{x}\in D} \sum_{k=1}^{m-1} \frac{|\eta_k(\mathbf{x})|^2}{\varrho_m(\mathbf{x})} \leq 2(m-1)$$

For systems $(\eta_k(\mathbf{x}))_{k=1}^{\infty}$, where for all $k \in \mathbb{N}$

$$\|\eta_k\|_{\ell_\infty(D)} \leq B,$$

we have

$$N_{K,\varrho_D}(m) \leq (m-1)B^2 \tag{12}$$

Theorem (Moeller, Ullrich' 21)

Let $\mathbf{y}^i$, $i = 1, \dots, n$, be i.i.d random sequences from ℓ_2. Let further $n \geq 3$, $r > 1$, $M > 0$ such that $\|\mathbf{y}^i\|_2 \leq M$ for all $i = 1, \dots, n$ almost surely and $\mathbb{E}\mathbf{y}^i \otimes \mathbf{y}^i = \mathbf{\Lambda}$ for $i = 1, \dots, n$ with $\|\mathbf{\Lambda}\|_{2\to 2} \leq 1$. Then

$$\mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\mathbf{y}^i \otimes \mathbf{y}^i - \mathbf{\Lambda}\right\|_{2\to 2} \geq F\right) \leq 2^{3/4} n^{1-r},$$

where $F := \max\left\{\frac{8r\log n}{n} M^2\kappa^2, \|\mathbf{\Lambda}\|_{2\to 2}\right\}$ and $\kappa = \frac{1+\sqrt{5}}{2}$.

Focus here on **infinite** random matrices, complements earlier results by Kämmerer, Ullrich, Volkmer, Tropp, Rauhut, Pajor, Mendelson, Oliveira...

$\mathbf{y}^i := \frac{1}{\sqrt{\varrho_m(\mathbf{x}^i)}}(e_m(\mathbf{x}^i), e_{m+1}(\mathbf{x}^i), \dots)^\top$, $i = 1, \dots, n$

$$\|\mathbf{y}^i\|_2^2 \leq \sup_{\mathbf{x}\in D}\sum_{k=m}^{\infty}\frac{|e_k(\mathbf{x})|^2}{\varrho_m(\mathbf{x})} \leq 2\sum_{k=m}^{\infty}\lambda_k =: M^2$$

$\mathbf{\Lambda} := \mathrm{diag}(\sigma_m^2, \sigma_{m+1}^2, \dots)$, $\|\mathbf{\Lambda}\|_{2\to 2} = \sigma_m^2$

Theorem (P., Ullrich' 21)

- $H(K)$ RKHS on a compact domain $D \subset \mathbb{R}^d$
- $K : D \times D \to \mathbb{C}$ continuous and bounded kernel
- ϱ_D finite Borel measure with full support on D
- $(\sigma_n)_{n=1}^{\infty}$, $\sigma_1 \geq \sigma_2 \geq \cdots$, singular values of $\mathrm{Id}\colon H(K) \to L_2(D, \varrho_D)$
- $m := \lfloor n/(c_1 r \log n) \rfloor$, $r > 1$

$$\sup_{\|f\|_{H(K)} \leq 1} \|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \leq c_3 \max\left\{ \frac{N_{K,\varrho_D}(m)}{m} \sum_{k \geq \lfloor m/2 \rfloor} \sigma_k^2, \sum_{k \geq \lfloor m/4 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\} \quad (13)$$

with probability larger than $1 - c_2 n^{1-r}$, where $\mathbf{X} = (\mathbf{x}^1, ..., \mathbf{x}^n)$ are drawn i.i.d. with respect to $\varrho_m(\cdot) d\varrho_D$, $N_{K,\varrho_D}(m) = \sup_{\mathbf{x} \in D} \sum_{k=1}^{m-1} \left|\sigma_k^{-1} e_k(\mathbf{x})\right|^2$, $(e_n(\mathbf{x}))_{n=1}^{\infty} \subset H(K)$.

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with probability larger than $1 - c_2 n^{1-r}$, where $\mathbf{X} = (\mathbf{x}^1, \ldots, \mathbf{x}^n)$ are drawn i.i.d. with respect to $\varrho_m(\cdot) d\varrho_D$, $N_{K,\varrho_D}(m) = \sup_{\mathbf{x} \in D} \sum_{k=1}^{m-1} \left|\sigma_k^{-1} e_k(\mathbf{x})\right|^2$, $(e_n(\mathbf{x}))_{n=1}^{\infty} \subset H(K)$.

$$c_2 = 3, \; c_3 = 4(\sqrt{2} + 8(1 + \sqrt{5})/\sqrt{2c_1})^2$$

Arbitrary ONS $(\eta_k(\mathbf{x}))_{k=1}^{\infty}$
$c_1 = 20, \; c_3 = 122$

$\|\eta_k\|_{\ell_\infty(D)} \leq 1, \; k \in \mathbb{N}$
$c_1 = 10, \; c_3 = 208$

$$\sup_{\|f\|_{H(K)}\le 1} \|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \le c_3 \max\left\{ \frac{N_{K,\varrho_D}(m)}{m} \sum_{k\ge\lfloor m/2\rfloor} \sigma_k^2, \sum_{k\ge\lfloor m/4\rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\}$$

If $N_{K,\varrho_D}(k) = \mathcal{O}(k)$, it holds

$$\sup_{\|f\|_{H(K)}\le 1} \|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \le C_{\varrho_D,K} \sum_{k\ge\lfloor m/4\rfloor} \sigma_k^2 \le C_{\varrho_D,K}\, a_{\lfloor m/4\rfloor}(\mathrm{Id}_{K,\infty})^2 \qquad (14)$$

F. Cobos, T. Kühn, W. Sickel Optimal approximation of multivariate periodic Sobolev functions in the sup-norm. *J. Funct. Anal.*, 270(11): 4196–4212, 2016.

$$m := \lfloor n/(c_1 r \log n) \rfloor, \quad r > 1$$

$$\varrho_m(\boldsymbol{x}) = \frac{1}{2}\left(\frac{1}{m-1}\sum_{k=1}^{m-1} |\eta_k(\boldsymbol{x})|^2 + \frac{1}{\sum_{k=m}^{\infty}\lambda_k}\sum_{k=m}^{\infty}|e_k(\boldsymbol{x})|^2\right)$$

$$\sup_{\|f\|_{H(K)}\le 1} \|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \le c_3 \max\left\{\frac{N_{K,\varrho_D}(m)}{m}\sum_{k\ge\lfloor m/2\rfloor}\sigma_k^2, \sum_{k\ge\lfloor m/4\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}\right\}$$

$$\varrho'_m(\boldsymbol{x}) = \frac{1}{2(m-1)}\sum_{k=1}^{m-1}|\eta_k(\boldsymbol{x})|^2 + \frac{1}{2}$$

$$\sup_{\|f\|_{H(K)}\le 1} \|f - \widetilde{S}_{\mathbf{X}}^m f\|_{\ell_\infty(D)}^2 \le c_3 \max\left\{\frac{N_{K,\varrho_D}(m)}{m}\sum_{k\ge\lfloor cm\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}, \sum_{k\ge\lfloor m/4\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}\right\}$$

$$m := \lfloor n/(c_1 r \log n) \rfloor, \quad r > 1$$

$$\varrho_m(\mathbf{x}) = \frac{1}{2}\Big(\frac{1}{m-1}\sum_{k=1}^{m-1}|\eta_k(\mathbf{x})|^2 + \frac{1}{\sum_{k=m}^{\infty}\lambda_k}\sum_{k=m}^{\infty}|e_k(\mathbf{x})|^2\Big)$$

$$\sup_{\|f\|_{H(K)}\le 1}\|f - \widetilde{S}_{\mathbf{X}}^m f\|^2_{\ell_\infty(D)} \le c_3 \max\Big\{\frac{N_{K,\varrho_D}(m)}{m}\sum_{k\ge\lfloor m/2\rfloor}\sigma_k^2, \sum_{k\ge\lfloor m/4\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}\Big\}$$

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m^* is the largest number such that $N_{K,\varrho_D}(m) \le n/(10 r \log n)$

$$\sup_{\|f\|_{H(K)}\le 1}\|f - S_{\mathbf{X}}^{m^*} f\|^2_{\ell_\infty(D)} \le C \sum_{k\ge\lfloor m^*/4\rfloor}\frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \qquad (15)$$

$$\mathbf{L}_{n,m} := \mathbf{L}_{n,m}(\mathbf{X}) = \begin{pmatrix} \eta_1(\mathbf{x}^1) & \eta_2(\mathbf{x}^1) & \cdots & \eta_{m-1}(\mathbf{x}^1) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1(\mathbf{x}^n) & \eta_2(\mathbf{x}^n) & \cdots & \eta_{m-1}(\mathbf{x}^n) \end{pmatrix}$$

Above approach requires $n = \mathcal{O}(m \log m)$ samples.

We "shrink" the matrix $\mathbf{L}_{n,m}$ to $\mathcal{O}(m)$ lines applying a modification of the Weaver sub-sampling strategy.

$$\widetilde{S}^m_{\mathbf{X}}, \ \mathbf{X} = \{\mathbf{x}^1, \dots, \mathbf{x}^n\} \quad \Longrightarrow \quad \widetilde{S}^m_J, \ \#J = \mathcal{O}(m), \ (\mathbf{x}^i)_{i \in J} \subset \mathbf{X}$$

N. Nagel, M. Schäfer, T. Ullrich A new upper bound for sampling numbers. *Found. Comp. Math.*, 2021.

S. Nitzan, A. Olevskii, A. Ulanovskii Exponential frames on unbounded sets. *Proc. Amer. Math. Soc.*, 144(1):109–118, 2016.

I. Limonova, V. N. Temlyakov On sampling discretization in L_2. *arXiv: math/2009.10789v1*, 2020.

Theorem (Nitzan, Olevskii, Ulanovskii' 16, Limonova, Temlyakov' 20, Nagel, Schäfer, Ullrich' 20)

Let $k_1, k_2, k_3 > 0$ and $\mathbf{u}_1, ..., \mathbf{u}_n \in \mathbb{C}^m$ with $\|\mathbf{u}_i\|_2^2 \leq k_1 \frac{m}{n}$ for all $i = 1, ..., n$ and

$$k_2 \|\mathrm{w}\|_2^2 \leq \sum_{i=1}^{n} |\langle \mathbf{w}, \mathbf{u}_i \rangle|^2 \leq k_3 \|\mathbf{w}\|_2^2 \,, \quad \mathbf{w} \in \mathbb{C}^m$$

$\Longrightarrow \quad \exists J \subseteq \{1, \ldots, n\},\ \#J \leq C_1 m$:

$$C_2 \cdot \frac{m}{n} \|\mathbf{w}\|_2^2 \leq \sum_{i \in J} |\langle \mathbf{w}, \mathbf{u}_i \rangle|^2 \leq C_3 \cdot \frac{m}{n} \|\mathbf{w}\|_2^2, \quad \mathbf{w} \in \mathbb{C}^m$$

More precisely, we can choose

$$C_1 = 1642 \frac{k_1}{k_2}\,, \quad C_2 = (2 + \sqrt{2})^2 k_1\,, \quad C_3 = 1642 \frac{k_1 k_3}{k_2}$$

in case $\frac{n}{m} \geq 47 \frac{k_1}{k_2}$. In the regime $1 \leq \frac{n}{m} < 47 \frac{k_1}{k_2}$ one may put $C_1 = 47 \frac{k_1}{k_2}$, $C_2 = k_2$, $C_3 = 47 \frac{k_1 k_3}{k_2}$.

Theorem (P., Ullrich' 21)

For $\mathrm{Id}\colon H(K) \to \ell_\infty(D)\,,\ \exists b, c_4, c_5, c_6 > 0$:

$$g_{\lfloor bm \log m \rfloor}(\mathrm{Id})^2 \le c_3 \max\left\{\frac{N_{K,\varrho_D}(m)}{m} \sum_{k \ge \lfloor m/2 \rfloor} \sigma_k^2\,, \sum_{k \ge \lfloor m/4 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}\right\}$$

$$g_m(\mathrm{Id})^2 \le c_4 \max\left\{\frac{N_{K,\varrho_D}(m) \log m}{m} \sum_{k \ge \lfloor c_5 m \rfloor} \sigma_k^2\,, \sum_{k \ge \lfloor c_5 m \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k}\right\}$$

The measure ϱ_D is at our disposal.

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The measure ϱ_D is at our disposal.

If $N_{K,\varrho_D}(k) = \mathcal{O}(k)$ we obtain

$$g_m(\mathrm{Id}) \le C_{\varrho_D,K} \min\{ a_{\lfloor m/(c_6 \log m) \rfloor}(\mathrm{Id}) \,,\ \sqrt{\log m} \cdot a_{\lfloor c_5 m \rfloor}(\mathrm{Id}) \} \qquad (16)$$

The power of standard information

$\mathrm{Id}\colon H(K) \to F$

$$q_F^{\mathrm{lin}} := \sup\left\{q \geq 0\colon \quad \lim_{n\to\infty} n^q a_n(\mathrm{Id}) = 0\right\}$$
$$q_F^{\mathrm{std}} := \sup\left\{q \geq 0\colon \quad \lim_{n\to\infty} n^q g_n(\mathrm{Id}) = 0\right\}$$

$$F = L_2(D, \varrho_D) \implies q_{2,\varrho_D}^{\mathrm{lin}} := q_{L_2(D,\varrho_D)}^{\mathrm{lin}}, \; q_{2,\varrho_D}^{\mathrm{std}} := q_{L_2(D,\varrho_D)}^{\mathrm{std}}$$
$$F = \ell_\infty(D) \implies q_\infty^{\mathrm{lin}} := q_{\ell_\infty(D)}^{\mathrm{lin}}, \qquad q_\infty^{\mathrm{std}} := q_{\ell_\infty(D)}^{\mathrm{std}}$$

For $a_n(\mathrm{Id})$ one can take $F = L_\infty(D)$

E. Novak, H. Woźniakowski Tractability of multivariate problems. Volume III: Standard information for operators, volume 18 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2012.

Assumptions

(i) $N_{K,\varrho_D}(k) = \mathcal{O}(k^u)$;

(ii) $\exists p > 1/2\,,\ C_2 > 0\colon \qquad \sigma_j \le C_2 j^{-p},\ j = 1, 2, \dots .$

$$u := \inf\left\{u\colon\ N_{K,\varrho_D}(k) = \mathcal{O}(k^u)\right\},$$

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F. Kuo, G. W. Wasilkowski, H. Woźniakowski

- Multivariate L_∞ approximation in the worst case setting over reproducing kernel Hilbert spaces. *J. Approx. Theory*, 152(2):135–160, 2008.
- On the power of standard information for multivariate approximation in the worst case setting. *J. Approx. Theory*, 158(1):97–125, 2009.

(i') $\exists C_1 > 0 \qquad \|\eta_j\|_{\ell_\infty(D)} \le C_1,\ j = 1, 2, \ldots$

$$N_{K,\varrho_D}(k) = \sup_{\mathbf{x}\in D} \sum_{i=1}^{k-1} |\eta_i(\mathbf{x})|^2 = \mathcal{O}(k),\ \ \eta_i(\mathbf{x}) = \sigma_i^{-1} e_i(\mathbf{x})$$

The power of standard information

(i) $N_{K,\varrho_D}(k) = \mathcal{O}(k^u)$;

(i') $\exists C_1 > 0$: $\|\eta_j\|_{\ell_\infty(D)} \le C_1$, $j = 1, 2, \ldots$

(ii) $\exists p > 1/2$, $C_2 > 0$: $\sigma_j \le C_2 j^{-p}$, $j = 1, 2, \ldots$

$$\text{(ii)} \quad \Longrightarrow \quad q^{\text{lin}}_{2,\varrho_D} = p$$

$$\text{(i')} \,/\, \text{(i)} \;\; \& \;\; \text{(ii)} \quad \Longrightarrow \quad q^{\text{lin}}_{\infty} = q^{\text{lin}}_{2,\varrho_D} - 1/2 = p - 1/2$$

$$\text{(i')} \;\; \& \;\; \text{(ii)} \quad \Longrightarrow \quad q^{\text{std}}_{\infty} \in \left[\frac{2p}{2p+1} \left(p - \frac{1}{2} \right), p - \frac{1}{2} \right]$$

Corollary (P., Ullrich' 21)

(i) & (ii), $2p > u$

$$q^{\text{std}}_{\infty} \ge p - u/2$$

In case $u = 1$ we have

$$q^{\text{std}}_{\infty} = q^{\text{lin}}_{\infty} = p - 1/2$$

Examples. Sobolev type spaces.

H^w, $w(\boldsymbol{k}) > 0$, $\boldsymbol{k} \in \mathbb{Z}^d$

$$\|f\|^2_{H^w(\mathbb{T}^d)} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (w(\boldsymbol{k}))^2 |c_{\boldsymbol{k}}(f)|^2 < \infty \tag{17}$$

$c_{\boldsymbol{k}}(f) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathrm{x}) e^{-i\boldsymbol{k}\cdot\mathrm{x}} \mathrm{dx}, \quad \boldsymbol{k} \in \mathbb{Z}^d,\ \mathbb{T}^d = [0, 2\pi]^d$

$$H^w(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d) \iff \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (w(\boldsymbol{k}))^{-2} < \infty$$

F. Cobos, T. Kühn, W. Sickel Optimal approximation of multivariate periodic Sobolev functions in the sup-norm. *J. Funct. Anal.*, 270(11):4196–4212, 2016.

$$K_w(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{(w(\boldsymbol{k}))^2} \tag{18}$$

$H^w(\mathbb{T}^d) \hookrightarrow \ell_\infty(\mathbb{T}^d)$

Sampling recovery. Sobolev type spaces.

$$g_n(\mathrm{I}_w : H^w(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)) \le C \min\left\{ a_{\lfloor n/b \log n \rfloor}(\mathrm{I}_w)\,,\, \sqrt{\log n} \cdot a_{\lfloor cn \rfloor}(\mathrm{I}_w) \right\}$$

Note that

$$a_n(\mathrm{I}_w)^2 = \sum_{k \ge n+1} \tau_k^2\,,$$

where $(\tau_n)_{n\in\mathbb{N}} = (1/w(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$, $\quad \tau_1 \ge \tau_2 \ge \dots$

$$g_m(\mathrm{I}_w) \le C \log m \cdot a_{\lfloor m/\log m \rfloor}(\mathrm{I}_w) \qquad (19)$$

L. Kämmerer Multiple lattice rules for multivariate L_∞ approximation in the worst-case setting. *arXiv: math/1909.02290v1*, 2019.

L. Kämmerer, T. Volkmer Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices. *J. Approx. Theory*, 246:1–27, 2019.

$$H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2(\mathbb{T}^d) \colon \; \|f\|^2_{H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d)} = \sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^{d} (1 + |k_j|)^{2s} < \infty \right\}$$

$$H^{s,+}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \in L_2(\mathbb{T}^d) \colon \; \|f\|^2_{H^{s,+}_{\mathrm{mix}}(\mathbb{T}^d)} = \sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^{d} \left(1 + |k_j|^2\right)^{s} < \infty \right\}$$

T. Kühn, W. Sickel, T. Ullrich Approximation of mixed order Sobolev functions on the d-torus: asymptotics, preasymptotics and d-dependence. *Constr. Approx.*, 42:353–398, 2015.

$$\mathrm{Id} \colon H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d), \quad \sigma_n \lesssim_{s,d} n^{-s} (\log n)^{s(d-1)}$$

Theorem (P., Ullrich' 21)

Let $s > 1/2$, $r > 1$, $m = \lfloor n/(c_1 r \log n) \rfloor$, $c_1 > 0$. *Then*

$$\sup_{\|f\|_{H^s_{\mathrm{mix}}(\mathbb{T}^d)} \le 1} \|f - S^m_{\mathbf{X}} f\|_{L_\infty(\mathbb{T}^d)} \lesssim_{s,d,r} n^{-s+1/2} (\log n)^{sd-1/2} \qquad (20)$$

is true as $n \to \infty$ *with probability larger than* $1 - 3n^{1-r}$.

$$g_{\lfloor bm\log m\rfloor}(\mathrm{Id}\colon H^s_{\mathrm{mix}}(\mathbb{T}^d)\to L_\infty(\mathbb{T}^d)) \lesssim_{s,d} m^{-s+1/2}(\log m)^{s(d-1)} \qquad (21)$$

$$g_{\lfloor cn\rfloor}(\mathrm{Id}\colon H^s_{\mathrm{mix}}(\mathbb{T}^d)) \to L_\infty(\mathbb{T}^d)) \lesssim_{s,d} n^{-s+1/2}(\log n)^{s(d-1)+1/2} \qquad (22)$$

$$s(d-1)+1/2 < s(d-1) + s - 1/2 \quad \text{if} \quad s>1$$

For sparse grids

$$g_n(\mathrm{Id}\colon H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)) \asymp_{s,d} n^{-s+1/2}(\log n)^{(d-1)s}, \quad s>1/2$$

V. N. Temlyakov On approximate recovery of functions with bounded mixed derivative.
J. Complexity, 9(1):41–59, 1993.

$$\sigma_n^{\#} \le \left(\frac{16}{3n}\right)^{\frac{s}{1+\log_2 d}}, \quad n \ge 6$$

T. Kühn New preasymptotic estimates for the approximation of periodic Sobolev functions. In *2018 MATRIX annals*, volume 3 of *MATRIX Book Ser.*, pages 97–112. Springer, Cham., 2020.

Theorem (P., Ullrich' 21)

- $s > (1+\log_2 d)/2$, $\beta := 2s/(1+\log_2 d) > 1$, $r > 1$
- $n \in \mathbb{N}$, $n \ge 3$, $m \in \mathbb{N}$ $m = \lfloor n/(10r\log n) \rfloor$

$$\sup_{\|f\|_{H^{s,\#}_{\text{mix}}(\mathbb{T}^d)} \le 1} \|f - S^m_{\mathbf{X}} f\|^2_{L_\infty(\mathbb{T}^d)} \le 832 \left(\frac{16}{3}\right)^{\beta} \frac{\beta}{\beta-1} \left(\frac{m}{4} - 1\right)^{-\beta+1} \quad (23)$$

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$$\sigma_n^{+} \leq \left(\frac{C(d)}{n}\right)^{\frac{s}{2(1+\log_2(d-1))}}, \tag{24}$$

$s > 0$, $d \geq 3$, $n \geq 2$, $C(d) = \left(1 + \frac{1}{d-1}\left(1 + \frac{2}{\log_2(d-1)}\right)\right)^{d-1}$

T. Kühn, W. Sickel, T. Ullrich How anisotropic mixed smoothness affects the decay of singular numbers for Sobolev embeddings. *J. Complexity*, 63, 2021.

General results

For $\mathrm{Id}\colon H(K) \to \ell_\infty(D)$, $\exists b, c_4, c_5, c_6 > 0$:

$$g_{\lfloor bm \log m \rfloor}(\mathrm{Id})^2 \le c_3 \max\left\{ \frac{N_{K,\varrho_D}(m)}{m} \sum_{k \ge \lfloor m/2 \rfloor} \sigma_k^2 \,,\, \sum_{k \ge \lfloor m/4 \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\}$$

$$g_m(\mathrm{Id})^2 \le c_4 \max\left\{ \frac{N_{K,\varrho_D}(m) \log m}{m} \sum_{k \ge \lfloor c_5 m \rfloor} \sigma_k^2 \,,\, \sum_{k \ge \lfloor c_5 m \rfloor} \frac{N_{K,\varrho_D}(4k)\sigma_k^2}{k} \right\}$$

A univariate example

$D = [-1, 1]$, the uniform measure dx on D

$$Af(x) = -((1-x^2)f')'$$

$$H(K_s) := \{f \in L_2(D) \colon \ A^{s/2}f \in L_2(D)\}, \quad s > 1$$

$$K_s(x,y) = \sum_{k\in\mathbb{N}} (1+(k(k+1))^s)^{-1}\mathcal{P}_k(x)\mathcal{P}_k(y)$$

$\mathcal{P}_k \colon D \to \mathbb{R}$, $k \in \mathbb{N}$, are $L_2(D)$-normalized Legendre polynomials $\mathcal{P}_k(x)$

- $(\eta_k)_{k=1}^{\infty} = (\mathcal{P}_k)_{k=1}^{\infty}$
- $(e_k)_{k=1}^{\infty} = ((1+(k(k+1))^s)^{-1/2}\mathcal{P}_k)_{k=1}^{\infty}$
- $\sigma_k = ((1+(k(k+1))^s)^{-1/2}$

$$N(m) = \mathcal{O}(m^2)$$

P. G. Nevai Orthogonal polynomials. *Mem. Amer. Math. Soc.*, 18(213), 1979.

$$g_n(\mathrm{Id}) \lesssim_s n^{-s+1}(\log n)^{\min\{s-1,1/2\}} \tag{25}$$

A univariate example

$L_2(D)$, Gauss points

C. Bernardi, Y. Maday Polynomial interpolation results in Sobolev spaces. *J. Comput. Appl. Math.*, 43(1):53–80, 1992.

$L_2(D)$, worst case error estimates with high probability

L. Kämmerer, T. Ullrich, T. Volkmer Worst case recovery guarantees for least squares approximation using random samples, *Constr. Approx.*, 2021.

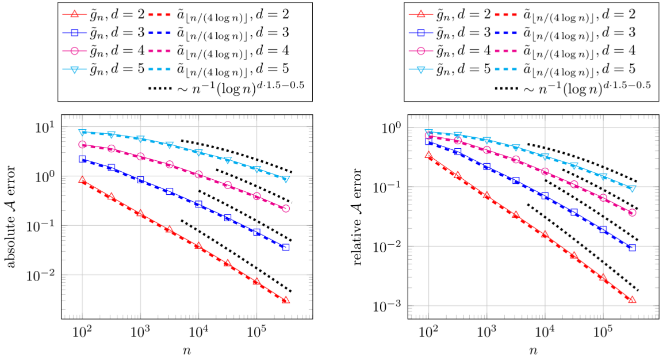
$$\sup_{\|f\|_{H(K_s)}\le 1} \|f - \widetilde{S}^m_{\mathbf{X}} f\|_{L_\infty(D)} \lesssim_s m^{-s+1} \lesssim n^{-s+1} (\log n)^{s-1}$$

$$\varrho_m(\mathbf{x}) = \frac{1}{2}\Big(\frac{1}{m-1}\sum_{k=1}^{m-1} |\eta_k(\mathbf{x})|^2 + \frac{1}{\sum_{k=m}^{\infty}\lambda_k}\sum_{k=m}^{\infty} |e_k(\mathbf{x})|^2\Big) \qquad (26)$$

$$\sup_{\|f\|_{H(K_s)}\le 1} \|f - \widetilde{S}^m_{\mathbf{X}} f\|_{L_\infty(D)} \lesssim_s n^{-s+3/2} (\log n)^{s-3/2}$$

$$\varrho'_m(\mathbf{x}) = \frac{1}{2(m-1)}\sum_{k=1}^{m-1} |\eta_k(\mathbf{x})|^2 + \frac{1}{2} \qquad (27)$$

$$\sup_{\|f\|_{H(K_s)}\le 1} \|f - S^m_{\mathbf{X}} f\|_{L_\infty(D)} \lesssim_s n^{-(s-1)/2} (\log n)^{(s-1)/2}$$



(a) absolute sampling and approximation eror

(b) relative sampling and approximation eror

Figure 1: Sampling errors (Wiener algebra norm) for non-periodic test function $f \in \tilde{H}^{3/2-\varepsilon}_{\text{mix}}([-1,1]^d)$, $m := \lfloor n/(4\log n)\rfloor$.

Made by T. Volkmer (Technische Universität Chemnitz)

Thank you
for your attention!