# Configurations and Erdős-Style Distance Problems

Jonathan Passant

Point Distriburtion Webinar

July 21, 2021

# Summary of Results

1. Rigid Graph Distances (joint with Iosevich)
    - We prove tight bounds for rotational congruence Erdős problem on Configurations.
    - We prove distance congruent bound for rigid graphs.
    - We provide a graphical Erdős conjecture.
    - Follows from Pinned-Distance Conjecture.

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    - We prove distance congruent bound for rigid graphs.
    - We provide a graphical Erdős conjecture.
    - Follows from Pinned-Distance Conjecture.

2. Hamiltonian Graph Distances
    - Expand Graphical Erdős conjecture to all graphs with Hamiltonian Paths.
    - Give a generalised incidence bound.
    - Iterative incidence bound.

# What are Distances?

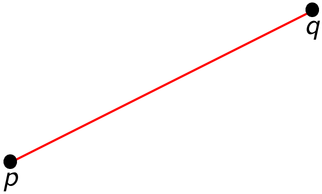


Figure: The distance $|p - q|$

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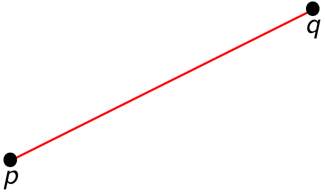


Figure: The distance $|p - q|$

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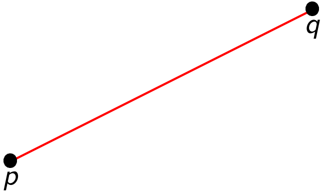



Figure: The distance $|p - q|$

Take two points $p$ and $q$ in the plane, their distance is denoted by

$$|p - q|$$

If $P$ is a set of points in the plane, we denote its set of distinct distances as

$$\Delta(P) = \{|p - q| : p, q \in P\}.$$

# Distinct Distances

Note that $\Delta(P)$ counts distinct distances.

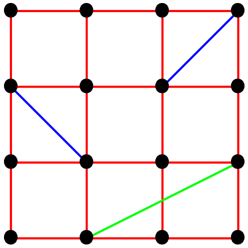
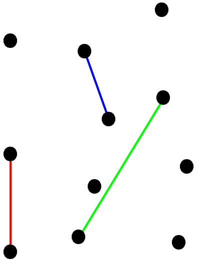


Figure: Lattice Distances



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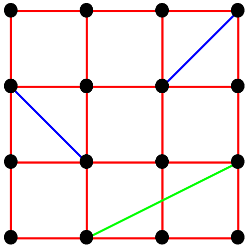

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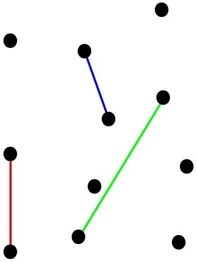

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**Note**

$$|\Delta(P)| \leqslant \binom{|P|}{2} \sim |P|^2$$

# Erdős Distinct Distance Problem

## Definition

Let $f(n)$ be the fewest distances a set of $n$ points in the plane makes.

$$f(n) = \min_{P \subseteq \mathbb{R}^2; |P| = n} |\Delta(P)|$$

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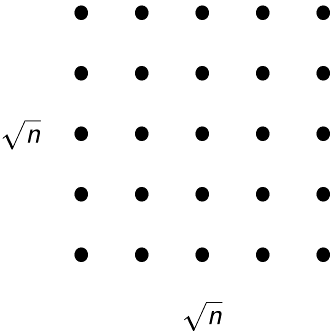


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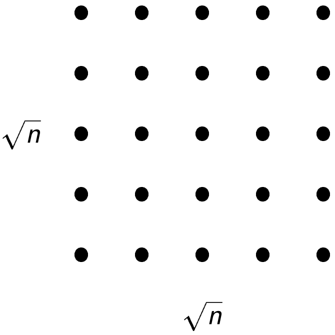


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So asymptotically

$$f(n) \lesssim \frac{n}{\sqrt{\log n}}.$$

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# Variants of the Distance Problem: Pinned Distances

## Pinned Erdős Conjecture

In all point sets $P$ of size $n$, there is a point from which the minimum number of distances are realised.

If one defines

$$f_{\text{pin}}(n) = \min_{P \subset \mathbb{R}^2; |P| = n} \max_{x \in P} \left| \{ |x - p| : p \in P \} \right|$$

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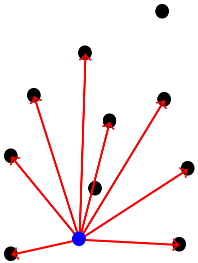

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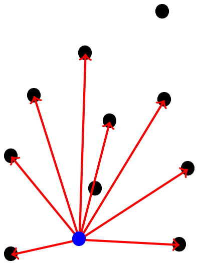

- Distances realised at one point.
- Conjecture the same as without pin.

# Pinned Distances Results

**Theorem (Erdős 1946)**

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**Theorem (Katz–Tardos 2004)**

$$n^{0.872\ldots} \lesssim f_{pin}(n) \lesssim \frac{n}{\sqrt{\log n}}.$$

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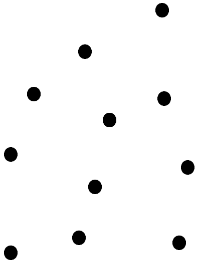

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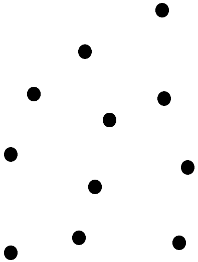

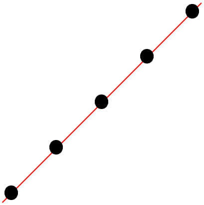

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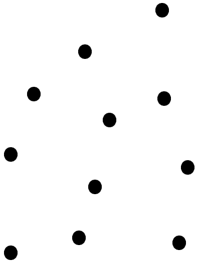
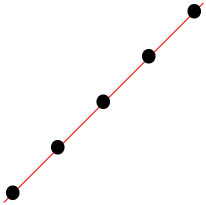
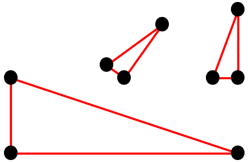




**Theorem (Solymosi–Tardos 2007)**

$$\frac{n^2}{\log(n)} \lesssim t_{sim}(n) \lesssim n^2.$$

# Congruent Triangles



Counted as one class of similar triangles

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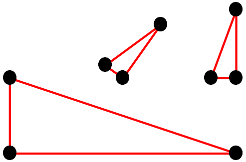

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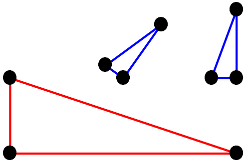

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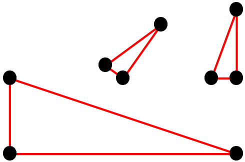

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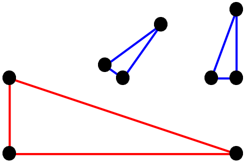

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- Congruence doesn't allow one to scale.

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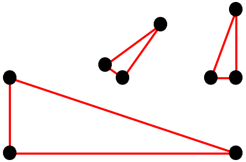

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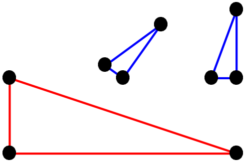


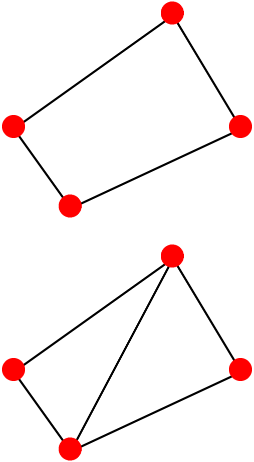
Counted as two classes of congruent triangles

- Congruence doesn't allow one to scale.

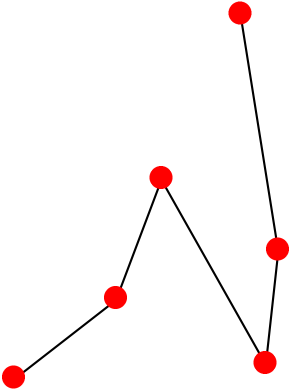
**Theorem (Rudnev 2012)**

$$n^2 \lesssim t_{cong}(n) \lesssim n^2.$$

# Larger Configurations

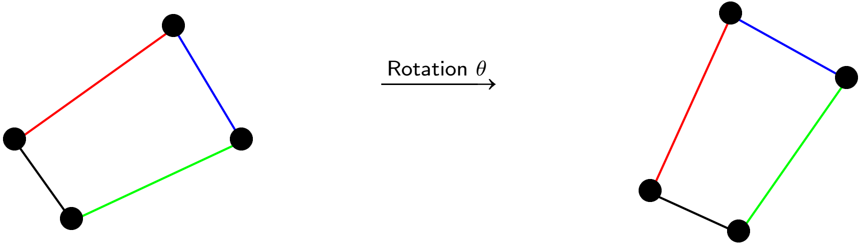


Four-point Configurations



Six-point Configuration

# Different Notions of Congruence



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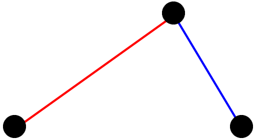

Same distances,

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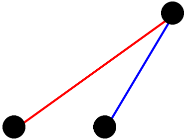

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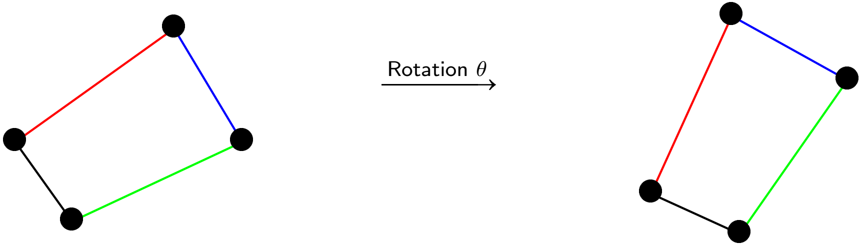


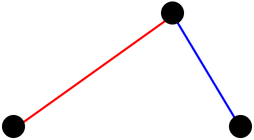

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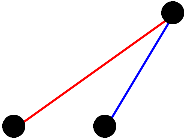

# Rotational Congruence: Relatively Simple

Let $M_k(P)$ be the number of congruence classes of non-singular $k$-tuples under the action of rigid motions.

$$m_k(n) = \min_{P \subseteq \mathbb{R}^2; |P| = n} |M_k(P)|$$

If $k = 3$ then $m_3(n) = t_{\text{cong}}(n)$, this is the congruent triangle problem.

**Theorem (Iosevich–P. 2018)**

*If $P$ is a set of $n$ points in the plane, $k \geqslant 4$ then*

$$n^{k-1} \lesssim m_k(n) \lesssim n^{k-1}.$$

# Proof: Counting Configuration Pairs

We let $v(t)$ be the number of $k$-tuples that realise the configuration $t \in M_k(P)$,

$$v(t) = \left|\{(p_1, \ldots, p_k) \in P^k : \overrightarrow{p} \text{ in cong. class } t\}\right|.$$

Then Cauchy-Schwarz tells us

$$|P|^{2k} = \left(\sum_{t \in M_k(P)} v(t)\right)^2 \leqslant |M_k(P)| \sum_t v^2(t).$$

Notice

$$\sum_t v^2(t) = \left|\{(\overrightarrow{p}, \overrightarrow{q}) \in P^{2k} : \overrightarrow{p} \text{ in same cong. class as } \overrightarrow{q}\}\right|.$$

# Congruent Pairs as Rigid Motions

- By Definition:

$$\overrightarrow{p} \text{ in same cong. class as } \overrightarrow{q} \iff \exists\theta \text{ such that } \overrightarrow{p} = \theta\overrightarrow{q}.$$

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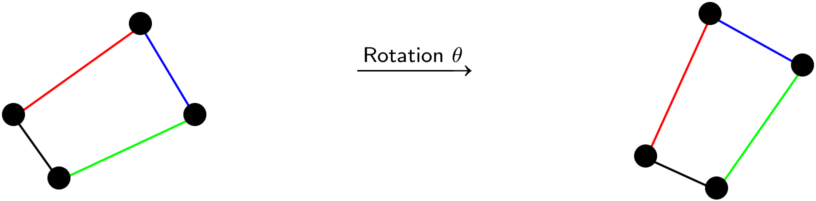


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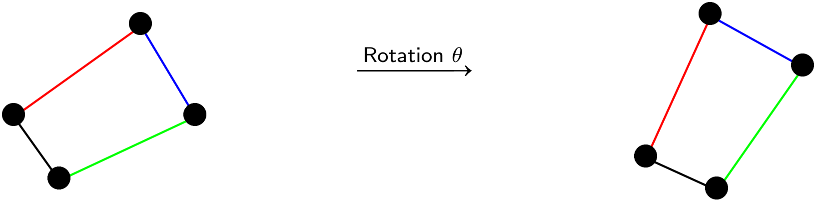


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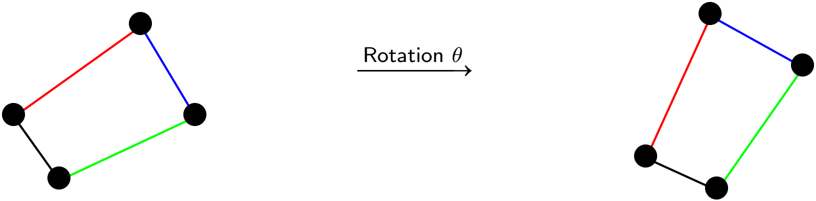
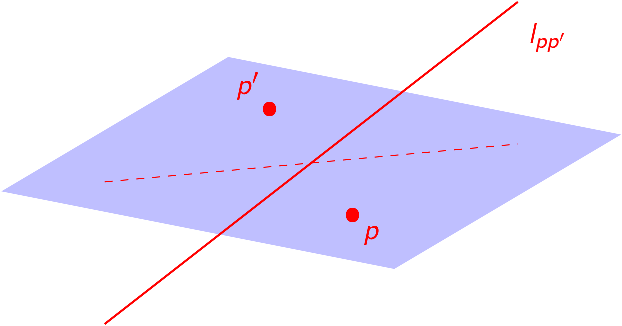



- We can see that $|P \cap \theta P| \geqslant k$ for any such motion.
- If $|P \cap \theta P| = r$ then $\theta$ contributes $\sim r^k$ pairs.
- If $R_{=r}(P) = \{\theta : |P \cap \theta P| = r\}$ then

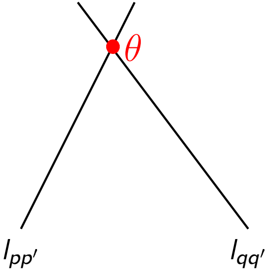
$$\sum_t v^2(t) \sim \sum_{r=k}^{|P|} r^k |R_{=r}(P)|.$$

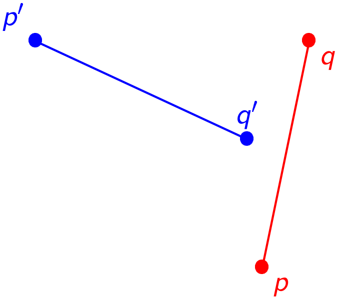
# Proof: Guth–Katz Lines

We use Guth-Katz lines, coming from rigid motions of the plane. Line $l_{pp'}$ represents all the rotations moving $p \xrightarrow{\theta} p'$.



# Guth–Katz Incidences




Intersections give pairs

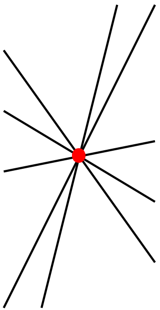
$$l_{pp'} \cap l_{qq'} \Leftrightarrow |p - q| = |p' - q'|$$

# Guth-Katz Bound

**Theorem (Guth–Katz 2010)**

*Let $L$ be a set of lines in $\mathbb{R}^3$ so that no more than $|L|^{1/2}$ lie in any plane or regulus. Then if $\mathcal{P}_r(L)$ are the points in $\mathbb{R}^3$ where at least $r$ such lines meet we have*

$$|\mathcal{P}_r(L)| \lesssim \frac{|L|^{3/2}}{r^2}.$$



$L = \{l_{pq} : p, q \in P\}$, so $|L| = |P|^2$. Thus,

$$|\mathcal{P}_r(L)| \lesssim \frac{|P|^3}{r^2}.$$

# Putting this together

We count pairs using rigid motions:

$$|P|^{2k} = \left(\sum_{t \in M_k(P)} v(t)\right)^2 \leqslant |M_k(P)| \sum_t v^2(t) \sim |M_k(P)| \sum_{r=k}^{|P|} r^k |\mathcal{P}_{=r}(L)|.$$

Using $|\mathcal{P}_{=r}(L)| = |\mathcal{P}_r(L)| - |\mathcal{P}_{r-1}(L)|$ and telescoping ($k > 2$),

$$\begin{aligned} |P|^{2k} &\lesssim |M_k(P)| \sum_{r=k}^{|P|} r^{k-1} |\mathcal{P}_r(L)| \lesssim |M_k(P)| \sum_{r=k}^{|P|} r^{k-3} |P|^3 \\ &\lesssim |M_k(P)||P|^{k+1}. \end{aligned}$$

# Proof Summary

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   - Congruence Definition $\rightarrow$ Uniform Rigid Motion

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    - Congruence Definition → Uniform Rigid Motion
3. Rigid motions to Incidences

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   - Congruence Definition → Uniform Rigid Motion
3. Rigid motions to Incidences
4. Incidence Bound

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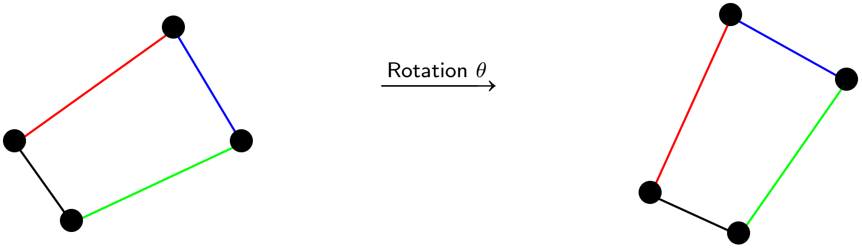

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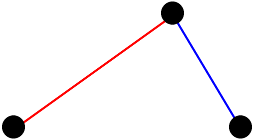

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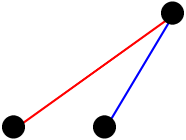

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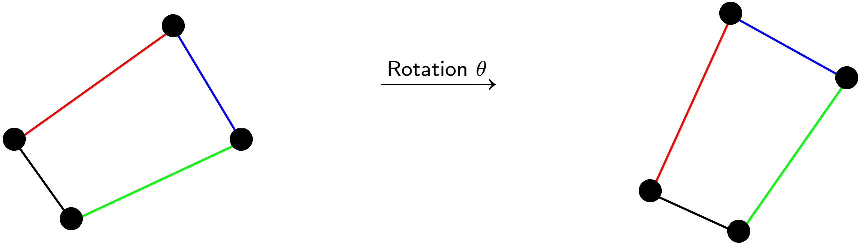


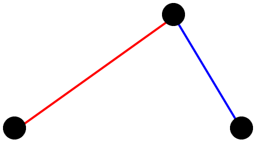

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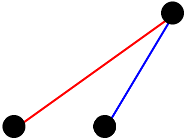
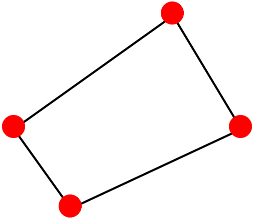


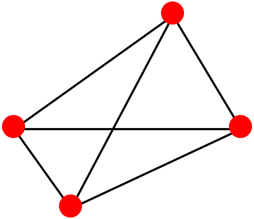
# Distance Congruence

We use a $k$-vertex connected graph $G$ to specify which edges we require to match.

$$\Delta_G(P) = \{(|p_i - p_j|)_{\{i,j\} \in E(G)} : p_i, p_j \in P\}.$$



$(\delta_1, \delta_2, \delta_3, \delta_4)$ counted in $\Delta_{C_4}(P)$



$(\delta_1, \ldots, \delta_6)$ counted in $\Delta_{K_4}(P)$

# Graphical Erdős Conjecture

Let

$$f_G(n) = \min_{P \subseteq \mathbb{R}^2; |P| = n} |\Delta_G(P)|$$

## Conjecture (Iosevich-P. 2018)

Suppose that $G$ is a connected graph on $k = O(1)$ vertices, then for all $\varepsilon > 0$ we have

$$n^{k-1-\varepsilon} \lesssim f_G(n) \lesssim n^{k-1}$$

- Upper bound obtained like similar triangles: on a line.
- $\varepsilon$ necessary for e.g. 3-chains.

# Rigidity Reduces to Rotational Congruence

**Theorem (Iosevich-P. 2018)**

Suppose that $G$ is a connected graph on $4 \leqslant k = O(1)$ vertices, then if $G$ is minimally infinitesimally rigid we have

$$\frac{n^{k-1}}{\log(n)} \lesssim f_G(n).$$

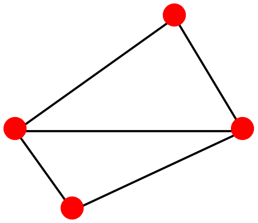


Figure: Rigid Graph

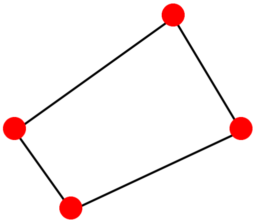


Figure: Non-rigid Graph

# Proof

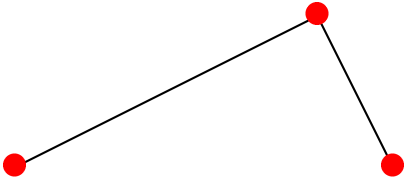
**Theorem (Chatzikonstantinou–Iosevich–Mkrtchyan–Pakianathan 2017)**

*If $G$ is minimally infinitesimally rigid, then there is a positive proportion of non-degenerate pairs $(\vec{p}, \vec{q})$ where the following are equivalent:*

- $|p_i - p_j| = |q_i - q_j|$ *for all* $\{i, j\} \in E(G)$.
- *There is a unique rigid motion* $\theta$ *such that* $\vec{p} = \theta\vec{q}$.

- We then run the same argument we saw for rotational congruence on this positive portion of pairs.
- Until the final part of the analysis works in any dimension.

# Non-Rigid Graphs



The 2-Chain

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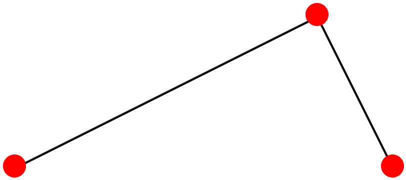

The 2-Chain

**Theorem (Rudnev 2019)**

Suppose that $G$ is a graph on $k = 3$ vertices, then

$$\frac{n^2}{\log^3 n} \lesssim f_G(n) \lesssim \frac{n^2}{\log n}$$

# New Non-Rigid Results

**Theorem (P. 2020)**

*Suppose that $G$ is a connected graph on $4 \leqslant k = O(1)$ vertices, then if $G$ has a Hamiltonian path we have*

$$\frac{n^{k-1}}{\log^{\frac{13}{2}(k-2)} n} \lesssim f_G(n) \lesssim n^{k-1}.$$

This is derived from the following result.

**Theorem (P. 2020)**

*Suppose that $C(k)$ is a chain on $4 \leqslant k = O(1)$ vertices then*

$$\frac{n^{k-1}}{\log^{\frac{13}{2}(k-2)} n} \lesssim f_{C(k)}(n) \lesssim \frac{n^{k-1}}{\log^{(k-1)/2} n}.$$

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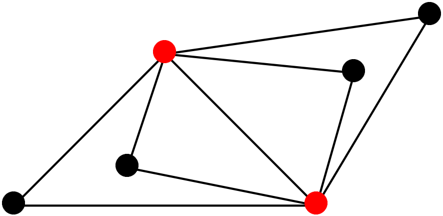

Figure: A rigid graph with no Hamiltonian path

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*Suppose that* $C(k)$ *is a chain on* $4 \leqslant k = O(1)$ *vertices then*

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- This resolves the graphical Erdős conjecture for a large class of graphs.
- Combining with the non-rigid result, star graphs seem the barrier to full resolution.

# Barrier to Full Conjecture

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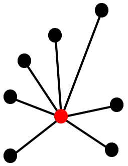

7-star

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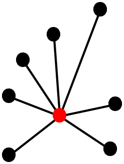

7-star

# Barrier to Full Conjecture

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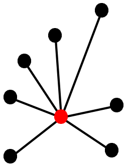
- 8-star and above beat Katz–Tardos.



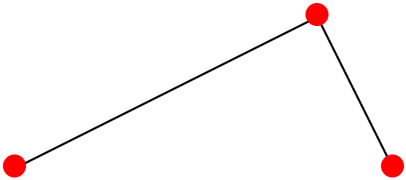
7-star

**Theorem (P. 2020, Stars give Pinned)**

*Establishing $n^{k-1-\varepsilon} \lesssim f_{k-star}(n)$ gives*

$$n^{\frac{k-1}{k}-\varepsilon} \lesssim f_{pin}(n).$$

# Rudnev's Proof



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Outline:

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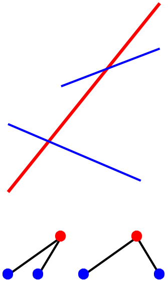

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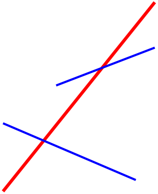
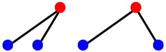
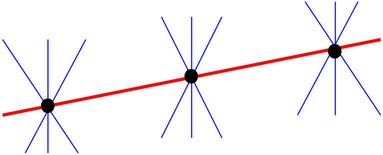
2. Incidence bound
   1. Double Partitioning $L_{t,\kappa} = L_t(P_\kappa)$
   2. Low weight: Use result of De Zeeuw
   3. High weight: Put in high degree surface, use result of Sharir–Solomon.





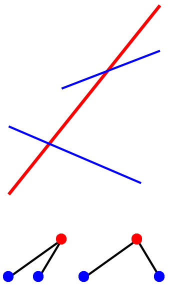


# Rudnev's Incident Bound

## Theorem (Rudnev 2019)

*Let $L$ be a set of lines with no more than $|L|^{1/2}$ in any plane or regulus. If $L_{t,\kappa}$ are those lines of $L$ that have $[t, 2t)$ points with $[\kappa, 2\kappa)$ lines of $L$ through them, then*

$$|L_{t,\kappa}| \lesssim \frac{|L|^2}{t^2\kappa^2}.$$



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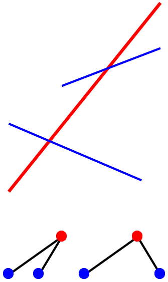

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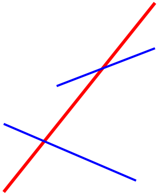

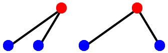

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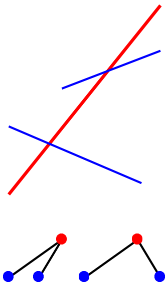

- Accounts for mass of intersections over a line.
- Gives an $L^2$ bound on line weights.
- $k$-star requires $L^k$ bound.

# Recall: Chain Result

**Theorem (P. 2020)**

*Suppose that $C(k)$ is a chain on $k+1$ vertices then*

$$\frac{n^{k-1}}{\log^{\frac{13}{2}(k-1)} n} \lesssim f_{C(k)}(n) \lesssim \frac{n^{k-1}}{\log^{(k-1)/2} n}.$$

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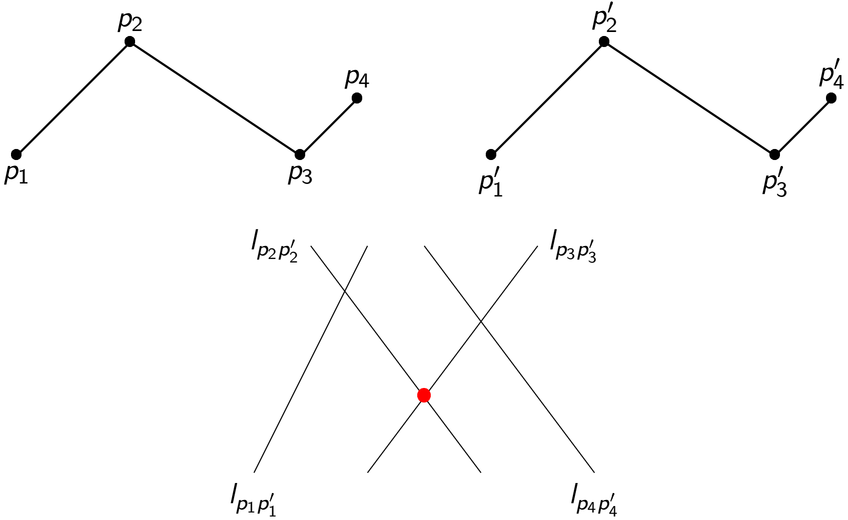
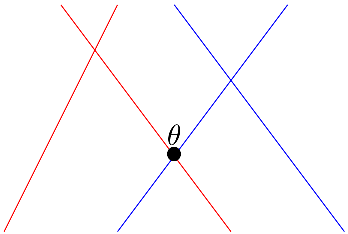
    - Generalise Rudnev: Allows one to use global structure
    - Iterative Incidence Bound

# Incidence set up for the 3-Chain

Energy set up requires we count pairs of 3-chains.



# Counting from the Central Point

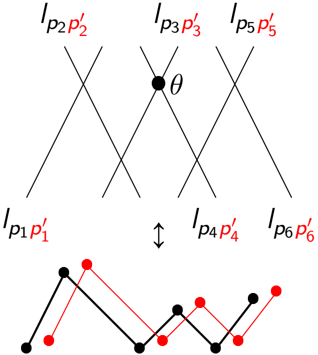



We look at points $\mathcal{P}_{t_1}(L_{t_2})$:

$\sim t_1$ lines through $\theta$, Each line has $\sim t_2$ lines crossing it.

$$|P|^8 \lessapprox |\Delta_{C(3)}(P)| \sum_{t_1,t_2} |\mathcal{P}_{t_1}(L_{t_2})| t_1^2 t_2^2$$

# Longer Chains Require More Variables




Pairs of 5-Chains

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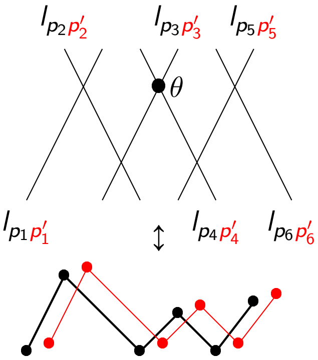


Pairs of 5-Chains

5 Chain thus requires 3 variables $t_1, t_2, t_3$.
Need,

$$|P_{t_1}(L_{t_2,t_3})| t_1^2 t_2^2 t_3^3 \lesssim |L|^{7/2}.$$

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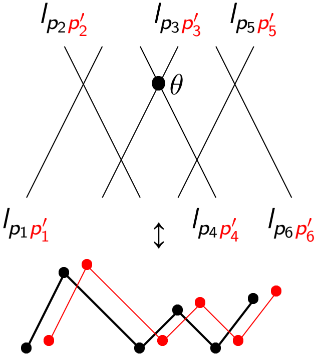


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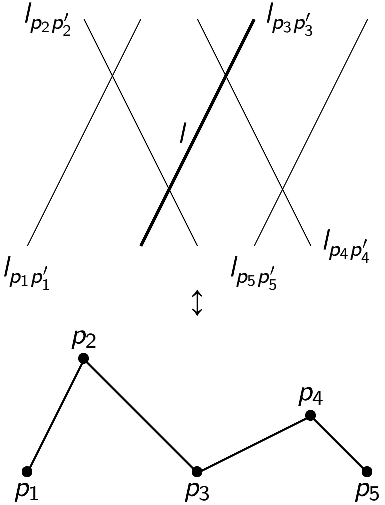
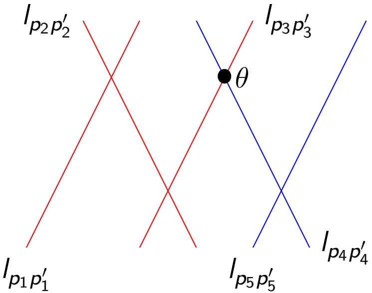
$k$-Chain needs $\frac{k+1}{2}$ variables. Require

$$|\mathcal{P}_{t_1}(L_{t_2,\dots,t_{(k+1)/2}})| t_1^2 \cdots t_{(k+1)/2}^2 \lessapprox |L|^{(k+2)/2}.$$




- 4-chain splits into red 5-chain and blue 3-chain pieces.
- Applying Cauchy–Schwarz and odd-chain result suffices.

# Apply Guth-Katz: Points to Line-Line

We want to estimate $|\mathcal{P}_{t_1}(L_{t_2,\ldots,t_{(k+1)/2}})|$.

**Theorem (Guth-Katz 2010)**

*Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$, let $s$ be a parameter so that $|\mathcal{L}|^{1/2} \leqslant s$ and no plane contains $s$ lines of $\mathcal{L}$. Let $\mathcal{P}_r$ be the set of points where at least $r$ of these lines meet. Then there is a constant $r_0$ such that for $r \geqslant r_0$ we have*

$$|\mathcal{P}_r| \lesssim \frac{|\mathcal{L}|^{3/2}}{r^2} + \frac{s|\mathcal{L}|}{r^3} + \frac{|\mathcal{L}|}{r}.$$

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$$|\mathcal{P}_{t_1;t_2,\dots,t_{(k+1)/2}}| \lesssim \frac{|\mathcal{L}_{t_2,\dots,t_{(k+1)/2}}|^{3/2}}{t_1^2} + \frac{|\mathcal{L}_{t_2,\dots,t_{(k+1)/2}}||L|^{1/2}}{t_1^3} + \frac{|\mathcal{L}_{t_2,\dots,t_{(k+1)/2}}|}{t_1}.$$

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- We use the global structure of $L$ applied to the nested subsets.
- So we have at most $s = |L|^{1/2}$ lines of $\mathcal{L}_{t_2,\dots,t_{(k+1)/2}}$ in any plane or regulus.

# Generalised Rudnev: Global Structure

## Theorem

*Suppose that $\mathcal{L}$ is a set of lines so that we have no more than $s \geqslant |\mathcal{L}|^{1/2}$ in any plane or regulus and no more that $s \geqslant |\mathcal{L}|^{1/2}$ lines concurrent. Then if $\mathcal{L}_{\kappa,t}$ are the lines of $\mathcal{L}$ that contain $t$ points with $\kappa$ lines of $\mathcal{L}$ through them then we have*

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- The second term comes from having worse control in linear components.

# Generalised Rudnev: Global Structure

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*Suppose that $\mathcal{L}$ is a set of lines so that we have no more than $s \geqslant |\mathcal{L}|^{1/2}$ in any plane or regulus and no more that $s \geqslant |\mathcal{L}|^{1/2}$ lines concurrent. Then if $\mathcal{L}_{\kappa,t}$ are the lines of $\mathcal{L}$ that contain $t$ points with $\kappa$ lines of $\mathcal{L}$ through them then we have*

$$|\mathcal{L}_{\kappa,t}| \lesssim \frac{|\mathcal{L}|s^2}{\kappa^2 t^2} + \frac{|\mathcal{L}|s\log(s)}{\kappa t}.$$

- The second term comes from having worse control in linear components.
- Proof is essentially Rudnev's with more bookkeeping. Over linear components an additional argument is required.

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- Proof is essentially Rudnev's with more bookkeeping. Over linear components an additional argument is required.
- Summing $kt \geqslant r$ provides the following Corollary.

# Rich Lines

## Corollary

If $\mathcal{L}$ is a set of lines with no more than $s \geqslant |\mathcal{L}|^{1/2}$ in any plane or regulus then if $\mathcal{L}_r$ are the lines with at least $r$ lines of $\mathcal{L}$ passing through them we have

$$|\mathcal{L}_r| \lesssim \frac{|\mathcal{L}|s^2 \log^2 |\mathcal{L}|}{r^2} + \frac{|\mathcal{L}|s \log^3 |\mathcal{L}|}{r}.$$

Recall:

$$\mathcal{L}_{t_2,\dots,t_{(k+1)/2}} \subseteq \mathcal{L}_{t_3,\dots,t_{(k+1)/2}} \subseteq \mathcal{L}_{t_{(k+1)/2}} \subseteq L.$$

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- If $s = |L|^{1/2}$ then $r \leqslant |L|^{1/2}$ first term, $r \geqslant |L|^{1/2}$ second term.
- We partition the $t_i$:

$$t_{i_a} \geqslant |L|^{1/2} \text{ and } t_{i_b} < |L|^{1/2}.$$

# Iterative Lemma

**Iterative Lemma**

Let $L$ be a line set in $\mathbb{R}^3$ with at most $|L|^{1/2}$ lines in any regulus or plane and at most $|L|^{1/2}$ lines concurrent.
If there are $\alpha$ many $t_{i_a}$ and $\beta$ many $t_{i_b}$ we have

$$|\mathcal{L}_{t_1,\ldots,t_{(k+1)/2}}| \lesssim \frac{|L||L|^{\alpha/2}|L|^{\beta}\log^{2\beta+3\alpha}|L|}{\prod_{a=1}^{\alpha} t_{i_a}\prod_{b=1}^{\beta} t_{i_b}^2}.$$

- As $P_{t_1;t_2,\ldots,t_{(k+1)/2}} \subseteq P_{t_1}(L)$, Guth-Katz gives

$$|P_{t_1;t_2,\ldots,t_{(k+1)/2}}|(t_1\cdots t_{(k+1)/2})^2 \lesssim |L|^{3/2}(t_2\cdots t_{(k+1)/2})^2.$$

- Playing these two off yields the result.

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4. Incidence Result:
   - Iterate line partitioning.
   - Use global structure to allow use of efficient results.
   - Generalised Rudnev
5. Carefully accounting of line weights and play off against Guth–Katz.

# Graphical Erdős Conjecture Summary

**Conjecture (Iosevich-P. 2018)**

Suppose that $G$ is a connected graph on $k = O(1)$ vertices, then for all $\varepsilon > 0$ we have

$$n^{k-1-\varepsilon} \lesssim f_G(n) \lesssim n^{k-1}$$

- Resolved for graphs with Hamiltonian paths and rigid graphs.
- Unresolved for star graphs.
- Logarithmic improvements can be made when loops are present.

Thank you