Intermittent symmetry breaking of the Agmon - Hörmander maximizers

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Point Distributions Webinar, 7th July 2021

Fourier Transform of Functions | ... and of Measures

$$\hat{F}(x) = \int_{\mathbb{R}^d} F(y)\, e^{i x \cdot y}\, dy. \qquad \hat{\mu}(x) = \int_{\mathbb{R}^d} e^{i x \cdot y}\, \mu(dy).$$

Plancherel Identity:

(functions only!)

$$\int_{\mathbb{R}^d} |\hat{F}(x)|^2 \frac{dx}{(2\pi)^d} = \int_{\mathbb{R}^d} |F(y)|^2\, dy.$$

$\sigma :=$ uniform measure on $\mathbb{S}^2$. $\qquad \hat{\sigma} = 4\pi \frac{\sin(r)}{r}.$

$$\int_{\mathbb{R}^3} |\hat{\sigma}(x)|^2 \frac{dx}{(2\pi)^3} = \frac{2}{\pi} |\mathbb{S}^2| \int_0^\infty \sin^2(r)\, dr = \infty !!!$$

However, $\frac{1}{R} \int_{\{|x| \le R\}} |\hat{\sigma}(x)|^2 \frac{dx}{(2\pi)^3} = \frac{2}{\pi} |\mathbb{S}^2| \frac{1}{R} \int_0^R \sin^2(r)\, dr = \frac{1}{\pi} |\mathbb{S}^2| + O\left(\frac{1}{R}\right).$

$(R \to \infty).$

S. Agmon & L. Hörmander, 1976. There is $C_d(R)>0$

such that
$$\frac{1}{R}\int_{\{|x|\leq R\}} |\widehat{f\sigma}(x)|^2 \frac{dx}{(2\pi)^d} \leq C_d(R)\int_{\mathbb{S}^{d-1}} |f|^2 d\sigma, \qquad \forall f\in L^2(\mathbb{S}^{d-1}).$$

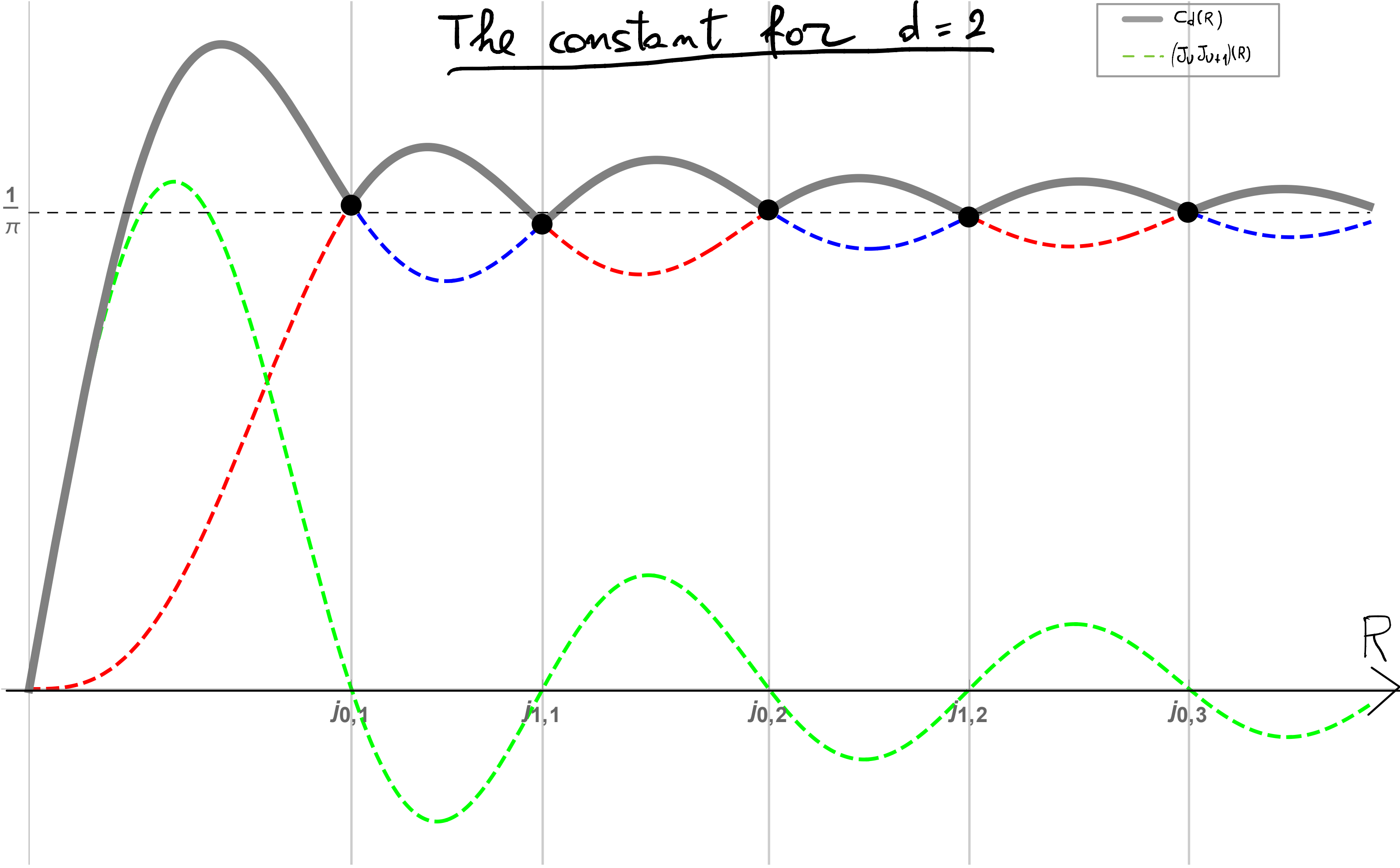
Moreover,
$$\lim_{R\to\infty} \frac{1}{R}\int_{\{|x|\leq R\}} |\widehat{f\sigma}(x)|^2 \frac{dx}{(2\pi)^d} = \frac{1}{\pi}\int_{\mathbb{S}^{d-1}} |f|^2 d\sigma.$$

$d\geq 2$

Our result, pt. 1. The best constant is

$$C_d(R) = \begin{cases} \frac{R}{2} J_\nu^2(R) - \frac{R}{2} J_{\nu-1}(R) J_{\nu+1}(R), & \text{if } J_\nu(R) J_{\nu+1}(R) > 0, \\ \\ \frac{R}{2} J_{\nu+1}^2(R) - \frac{R}{2} J_\nu(R) J_{\nu+2}(R), & \text{if } J_\nu(R) J_{\nu+1}(R) \leq 0. \end{cases}$$

$\nu := \frac{d}{2} - 1$, $\quad J_\alpha$ = Bessel function of the first kind.



The constant for d = 2
$C_d(R)$
$(J_\nu J_{\nu+1})(R)$
$\frac{1}{\pi}$
R
$j_{0,1}$
$j_{1,1}$
$j_{0,2}$
$j_{1,2}$
$j_{0,3}$

Our result, pt. 2.

Recall: $$C_d(R) = \max_{0 \neq f \in L^2(S^{d-1})} \frac{\frac{1}{R}\int_{\{|x|\leq R\}} |\widehat{f\sigma}(x)|^2 \frac{dx}{(2\pi)^d}}{\|f\|^2_{L^2(S^{d-1})}}.$$

$\omega = (\omega_1, \dots, \omega_d) \in S^{d-1}$

The max is attained if and only if:

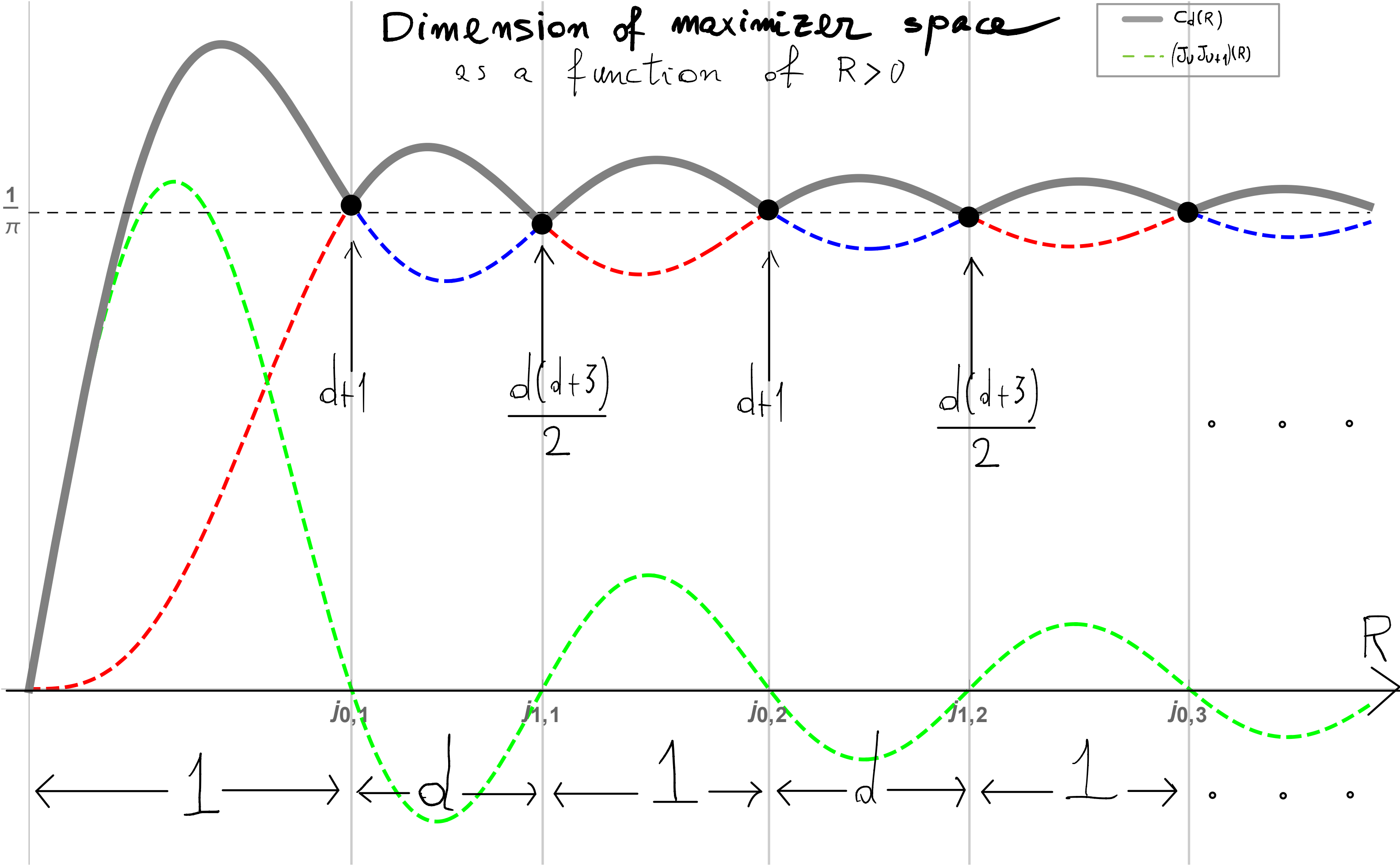
(i) when $(J_\nu J_{\nu+1})(R) > 0$, for $f(\omega) = c$.

(ii) when $(J_\nu J_{\nu+1})(R) < 0$, for $f(\omega) = \sum_{j=1}^{d} a_j \omega_j$.

(iii-a) when $J_\nu(R) = 0 \neq J_{\nu+1}(R)$, for $f(\omega) = c + \sum_{j=1}^{d} a_j \omega_j$.

(iii-b) when $J_\nu(R) \neq 0 = J_{\nu+1}(R)$, for $f(\omega) = H_2(\omega)$.

$c \in \mathbb{C}$, $(a_1, a_2, \dots, a_d) \in \mathbb{C}^d$, $H_2(\omega)$ a HARMONIC POLY, $\deg H_2 \leq 2$.


Dimension of maximizer space
as a function of R>0
$C_d(R)$
$(J_\nu J_{\nu+1})(R)$
$\frac{1}{\pi}$
$d+1$
$\frac{d(d+3)}{2}$
$d+1$
$\frac{d(d+3)}{2}$
R
$j_{0,1}$
$j_{1,1}$
$j_{0,2}$
$j_{1,2}$
$j_{0,3}$
1
d
1
d
1

Motivation. (or Why should we care?)

Ag.-Hör. is the simplest example of a Fourier restriction estimate, i.e.

$$\|\widehat{f\sigma}\|_{X(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{S}^{d-1})}.$$

Best constant is known for $X(\mathbb{R}^d) = L^{2k}(\mathbb{R}^d)$, $3 \leq d \leq 7$. $k \geq 2$

In all these cases, maximizers are constants,

i.e. fully symmetric.

This is our case for some values of R,

but not for all: **symmetry breaking**.

Proof We want $\boxed{C_d(R) = \max\limits_{f \neq 0} \dfrac{\frac{1}{R}\int_{\{|x|\leq R\}} |\widehat{f\sigma}(x)|^2 \frac{dx}{(2\pi)^d}}{\|f\|^2_{L^2(S^{d-1})}}}$.

Spherical harmonics: $Y_K(\omega)$ homogeneous harmonic poly deg. $= K$.

Decompose $f = \sum_{K=0}^{\infty} Y_K$ so that $\|f\|^2_{L^2(S^{d-1})} = \sum_{K=0}^{\infty} \|Y_K\|^2_{L^2(S^{d-1})}$.

KEY FACT: $$\frac{1}{R}\int_{\{|x|\leq R\}} |\widehat{f\sigma}(x)|^2 \frac{dx}{(2\pi)^d} = \sum_{K=0}^{\infty} \underbrace{\Lambda_{K,d}(R)}_{\geq 0} \|Y_K\|^2_{L^2(S^{d-1})}.$$

Thus,

$$C_d(R) = \max \frac{\sum_{k=0}^{\infty} \Lambda_{k,d}(R) \|Y_k\|^2}{\sum_{k=0}^{\infty} \|Y_k\|^2}$$

$$= \max \left\{ \Lambda_{0,d}(R), \Lambda_{1,d}(R), \Lambda_{2,d}(R), \ldots \right\}.$$

The Λ's are Lommel integrals: $\nu = \frac{d}{2} - 1.$

$$\Lambda_{k,d}(R) = \frac{1}{R} \int_0^R J_{\nu+k}^2(r)\, r\, dr = \frac{R}{2} J_{\nu+k}^2(R) - \frac{R}{2}\left(J_{\nu+k-1} J_{\nu+k+1}\right)(R).$$

Claim: $\boxed{\max\{\Lambda_{0,d}(R),\ \Lambda_{1,d}(R)\} \geq \Lambda_{k,d}(R)}$ for all k.

proof:

$$R\left(\Lambda_{k,d}(R) - \Lambda_{k+2,d}(R)\right) = \int_0^R \left[J_{\nu+k}^2(r) - J_{\nu+k+2}^2(r)\right] r\,dr$$

$$= \int_0^R \left(J_{\nu+k}(r) + J_{\nu+k+2}(r)\right)\left(J_{\nu+k}(r) - J_{\nu+k+2}(r)\right) r\,dr$$

(Bessel recursions) $$\cong \int_0^R J_{\nu+k+1}(r)\,\frac{d}{dr}\left(J_{\nu+k+1}(r)\right) dr \cong J_{\nu+k+1}^2(R) \geq 0.$$

We conclude that

$$\boxed{\Lambda_{0,d}(R) \geq \Lambda_{2,d}(R) \geq \Lambda_{4,d}(R) \geq \dots} \quad \boxed{\Lambda_{1,d}(R) \geq \Lambda_{3,d}(R) \geq \Lambda_{5,d}(R) \geq \dots}$$

$\square$

End of proof.

We saw that $C_d(R) = \max\{\Lambda_{0,d}(R), \Lambda_{1,d}(R)\}$.

Turns out that $\Lambda_{0,d}(R) - \Lambda_{1,d}(R) = (J_\nu J_{\nu+1})(R)$.

Conclusion: $$C_d(R) = \begin{cases} \Lambda_{0,d}(R), & (J_\nu J_{\nu+1})(R) > 0, \\ \Lambda_{1,d}(R), & (J_\nu J_{\nu+1})(R) < 0, \\ \Lambda_{0,1}(R) = \Lambda_{1,d}(R), & (J_\nu J_{\nu+1})(R) = 0. \end{cases}$$