Sharp estimates for the wave equation via the Penrose transform

by

Giuseppe Negro (Univ. Birmingham, UK)

Point Distributions Webinar

22 July 2020

Formal definitions. For $f:\mathbb{R}^d \to \mathbb{C}$,

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-i x\cdot\xi}\, dx.$$

Note that

$$\Delta f = \sum_{j=1}^{d} \partial_{x_j}^2 f = \mathcal{F}^{-1}\left[|\xi|^2 \hat{f}(\xi) \right].$$

Definition For $m:[0,\infty) \to \mathbb{C}$, we let

$$m(\sqrt{-\Delta}) f(x) = \mathcal{F}^{-1}\left[m(|\xi|)\hat{f}(\xi)\right].$$

Analogous definition for $m(\sqrt{-\Delta_{S^d}})g$, if $g: S^d \to \mathbb{C}$.

The operator $\mathrm{Re}\left(e^{it\sqrt{-\Delta}} f\right)$ is defined as above.

(This can be seen as Fourier extension operator to ).

Strichartz estimates (1977) (see also Tomás-Stein)

For each $d \geq 2$, there is $C_d > 0$ such that

$$\| \mathrm{Re}(e^{it\sqrt{-\Delta}} f) \|_{L^p(\mathbb{R}^{1+d})} \leq C_d \| (-\Delta)^{1/4} f \|_{L^2(\mathbb{R}^d)}, \quad p = 2\frac{d+1}{d-1}.$$

<u>Remark</u>. RHS is a "relativistic energy" (see the book "Analysis", Lieb & Loss).
LHS is a measure of "size".

<u>Theorem</u> (Foschi 2004). Let $f_*(x) = \left(\frac{2}{1+|x|^2} \right)^{\frac{d-1}{2}}$. Then, for $d=3$,

$$\max \left(\frac{\| \mathrm{Re}(e^{it\sqrt{-\Delta}} f) \|_{L^p}}{\| (-\Delta)^{1/4} f \|_{L^2}} \right)$$ is attained at $f = f_*$.

<u>Conjecture</u> The same should be true for all $d \geq 2$.

My results

Conjecture fails for d even. More precisely, (N. '18)

$$\left(f_* \text{ is a crit. point of } f \mapsto \frac{\|\mathrm{Re}(e^{it\sqrt{-\Delta}} f)\|_{L^p}^p}{\|(-\Delta)^{1/4} f\|_{L^2}^p} \right) \Longleftrightarrow \left(d \text{ is odd} \right).$$

If d is odd then f_* is a local maximizer (Gonçalves, N. '20)

<u>Remark</u>. Surprisingly, f_* is a crit. point and local maximizer of

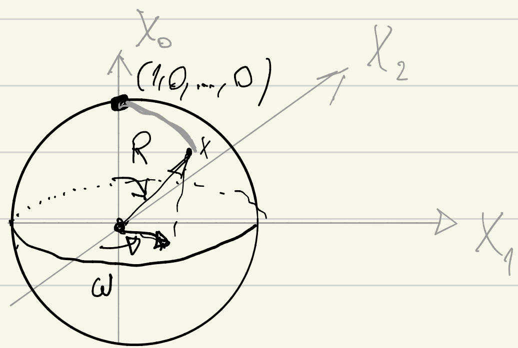
$$f \mapsto \frac{\|e^{it\sqrt{-\Delta}} f\|_{L^p}^p}{\|(-\Delta)^{1/4} f\|_{L^2}^p} \quad \text{for all } d \geq 2!$$

The Penrose transform: a classical tool of relativistic PDEs

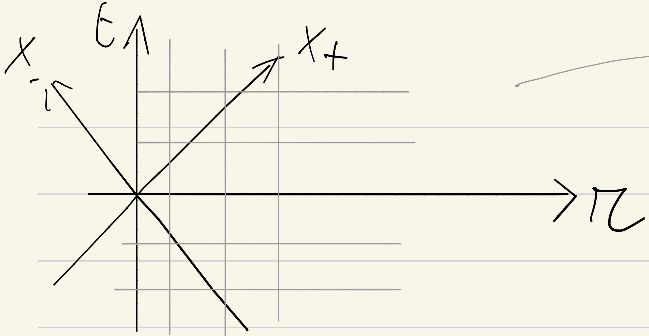
We want to map $\mathbb{R}^{1+d}$ into $[-\pi,\pi]\times S^d$, which is compact.
We introduce polar coordinates.

On $\mathbb{R}^{1+d}$: $t\in\mathbb{R}$, $x\in\mathbb{R}^d$ and $x=r\omega$
where $r=|x|$ and $\omega\in S^{d-1}$.

On $[-\pi,\pi]\times S^d$: $T\in[-\pi,\pi]$, $X=(X_0,X_1,\dots,X_d)\in S^d$.
(here $\sum_{j=0}^{d} X_j^2=1$).

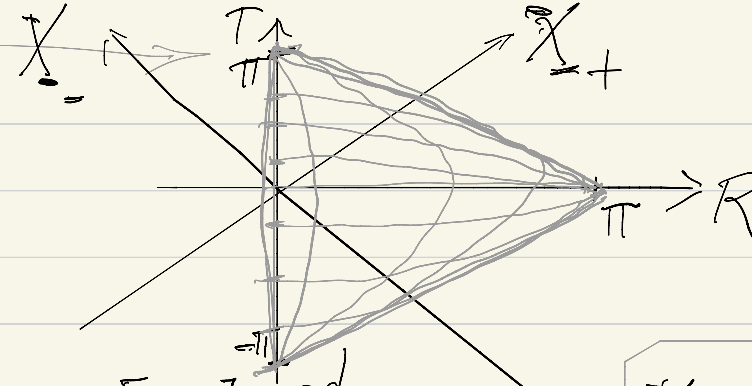


$$X=(\cos R,\ \sin(R)\omega)$$
where $R\in[0,\pi]$, $\omega\in S^{d-1}$.



$\mathbb{R}^{1+d}$

$$\boxed{x_\pm = t \pm r}$$



$[-\pi,\pi]\times S^d$

$$\boxed{X_\pm = \tfrac{1}{2}(T \pm R)}$$

Definition: Let $P : \mathbb{R}^{1+d} \longrightarrow [-\pi,\pi]\times S^d$ be defined by

$$X_\pm = \arctan(x_\pm).$$

Remark. P is NOT surjective.

$$P(\mathbb{R}^{1+d}) = \{ R \le \pi - |T| \}$$ is called "Penrose diagram" of $\mathbb{R}^{1+d}$.

Main property. Let $\Omega(t,r) = \dfrac{2}{\sqrt{1+(t-r)^2}\sqrt{1+(t+r)^2}}$, $\Omega_0(r) = \Omega\big|_{t=0}$

Definition. For $f:\mathbb{R}^d \to \mathbb{C}$ let $F: S^d \to \mathbb{C}$ be

$$F = \left(\Omega_0^{-\frac{d-1}{2}} f\right) \circ P_{t=0}^{-1}$$

F is the Penrose transform of f.

Then: (i) $\Omega^{-\frac{d-1}{2}} \cdot \mathrm{Re}\left(e^{it\sqrt{-\Delta}} f\right) \circ P^{-1} = \mathrm{Re}\left(e^{iT\sqrt{-\Delta_{S^d} + \left(\frac{d-1}{2}\right)^2}} F\right)$.

(ii) $\|(-\Delta)^{1/4} f\|_{L^2(\mathbb{R}^d)} = \left\|\left(-\Delta_{S^d} + \left(\frac{d-1}{2}\right)^2\right)^{1/4} F\right\|_{L^2(S^d)}$.

Sketch of proof: Everything follows from

$$P^*\left(ds^2_{[-\pi,\pi]\times S^d}\right) = \Omega^2 ds^2_{\mathbb{R}^{1+d}},$$

which says that P is a CONFORMAL MAP. Indeed, this implies that

$$(\partial_t^2 - \Delta)u = \Omega^{\frac{d+3}{2}} \cdot \left(\partial_T^2 - \Delta_{S^d} + \left(\frac{d-1}{2}\right)^2\right)\left((\Omega^{-\frac{d-1}{2}}u)\circ P\right)$$

Property (i) follows.

Property (ii) is also related to conformal theory (keyword: "conformal Fractional Laplacian").

We can finally prove the original result.

Foschi's conjectured maximizer is $f_* = \left(\frac{2}{1+r^2}\right)^{\frac{d-1}{2}}$. Its Penrose transform is $F_* = 1$.

Lemma (Lagrange multiplier) Let $Sf = \mathrm{Re}\,(e^{it\sqrt{-\Delta}} f)$.

$\left[f_* \text{ is a critical point for } \frac{\|Sf\|_{L^p}^p}{\|(-\Delta)^{1/4} f\|_{L^2}^p} \right] \Longleftrightarrow \left[\text{There is } \mu \in \mathbb{R} \text{ s.t. } \partial_{\varepsilon=0} \|Sf_* + \varepsilon Sf\|_p^p = \mu \langle f_* | f \rangle \right]$

where $\langle f_* | f \rangle = \mathrm{Re} \int (-\Delta)^{1/4} f_* \, \overline{(-\Delta)^{1/4} f} \, dx$.

Corollary

$[f_* \text{ is a crit. pt.}] \Longleftrightarrow \left[\iint_{\mathbb{R}^{1+d}} |Sf_*|^{p-2} Sf_* \, Sf_\perp \, dt\, dx = 0, \ \forall f_\perp \text{ such that } \langle f_\perp | f_* \rangle = 0 \right]$.

Remark. Since $F_* = 1$, $Sf_* \cong \mathrm{Re}\left(e^{iT\sqrt{-\Delta_{S^d} + \left(\frac{d-1}{2}\right)^2}}\, 1\right) = \mathrm{Re}\left(e^{iT\frac{d-1}{2}}\right)$.

By Penrose, we are led to study

$$\iint_{P(\mathbb{R}^{1+d})} \left|\cos\left(\frac{d-1}{2}T\right)\right|^{p-2} \cos\left(\frac{d-1}{2}T\right) \mathrm{Re}\left(e^{iT\sqrt{-\Delta_{S^d} + \left(\frac{d-1}{2}\right)^2}}\, F_1\right) dT\, d\sigma,$$

where $d\sigma$ is the measure on S^d. Problem: $P(\mathbb{R}^{1+d})$ is NOT a cartesian product!

where $d\sigma$ = surface measure on $\mathbb{S}^d$, $U_\perp = \mathrm{Re}\left(e^{iT\sqrt{-\Delta_{\mathbb{S}^d} + \left(\frac{d-1}{2}\right)^2}} F_\perp\right)$.

Recall: <u>Spherical harmonics</u>. We let $\{Y_{\ell,m}\}$ denote a ONB of $L^2(\mathbb{S}^d)$ s.t. $-\Delta_{\mathbb{S}^d} Y_{\ell,m} = \ell(\ell+d-1) Y_{\ell,m}$.

For $F \in L^2(\mathbb{S}^d)$, $\hat{F}(\ell,m) = \int_{\mathbb{S}^d} F\, \overline{Y_{\ell,m}}\, d\sigma$.

<u>Remark</u> We have: $U_\perp = \mathrm{Re}\, e^{iT\sqrt{-\Delta_{\mathbb{S}^d} + \left(\frac{d-1}{2}\right)^2}} F = \mathrm{Re} \sum_{\ell\, m} e^{iT\left(\ell + \frac{d-1}{2}\right)} \hat{F}(\ell,m)$,

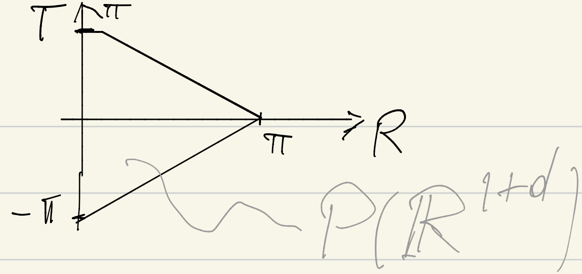
because $\sqrt{-\Delta_{\mathbb{S}^d} + \left(\frac{d-1}{2}\right)^2}\, Y_{\ell,m}(X) = \sqrt{\ell^2 + \ell(d-1) + \left(\frac{d-1}{2}\right)^2}\, Y_{\ell,m}(X)$

$= \left(\ell + \frac{d-1}{2}\right) Y_{\ell,m}(X)$.

Recall: $X \in \mathbb{S}^d$

Important! $\frac{d-1}{2}$ is integer IFF d is odd.

Recall we are on



We consider $U_\perp = \mathrm{Re} \sum_{\ell, m} e^{iT\left(\ell + \left(\frac{d-1}{2}\right)\right)} \hat{F}_\perp(\ell, m)\, Y_{\ell, m}(\underline{X})$.

We recognize the symmetries

(i) $U_\perp(T + 2\pi, \underline{X}) = U_\perp(T, \underline{X})$, d odd.

(ii) $U_\perp(T + 4\pi, \underline{X}) = U_\perp(T, \underline{X})$, d even.

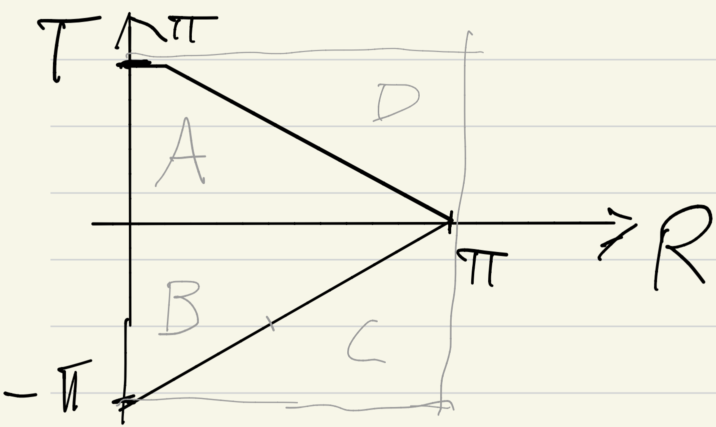
(iii) $U_\perp(T + \pi, -\underline{X}) = (-1)^{\frac{d-1}{2}} U_\perp(T, \underline{X})$, d odd.

Use that $Y_{\ell m}(-\underline{X}) = (-1)^{\ell} Y_{\ell m}(\underline{X})$

Corollary. If d is Odd,

$$I = \iint_{P(\mathbb{R}^{1+d})} \left|\cos\left(\tfrac{d-1}{2}T\right)\right|^{p-2} \cos\left(\tfrac{d-1}{2}T\right) U_\perp \, dT\, d\sigma = \frac{1}{2}\iint_{[-\pi,\pi]\times S^d} \left|\cos\left(\tfrac{d-1}{2}T\right)\right|^{p-2}\cos\left(\tfrac{d-1}{2}T\right) U_\perp \, dT\, d\sigma.$$

proof



$$U_\perp(T+2\pi, X) = U_\perp(T, X);$$
$$U_\perp(T+\pi, -X) = (-1)^{\frac{d-1}{2}} U_\perp(T, X).$$

The LHS is $\iint_A + \iint_B$.

The RHS is $\iint_A + \iint_B + \iint_C + \iint_D$.

By the symmetries, $\iint_A = \iint_C$ and $\iint_B = \iint_D$. This FAILS for d even!

We conclude the proof in the odd d case.

$$\frac{1}{2}\int_{-\pi}^{\pi}\int_{S^d}\left|\cos\left(\frac{d-1}{2}T\right)\right|^{p-2}\cos\left(\frac{d-1}{2}T\right)U_\perp\, dT\, d\sigma \overset{\text{Fubini}}{=}$$

$$\frac{1}{2}\int_{-\pi}^{\pi}\left|\cos\left(\frac{d-1}{2}T\right)\right|^{p-2}\cos\left(\frac{d-1}{2}T\right)\left(\int_{S^d}U_\perp\, d\sigma\right)dT$$

and $\int U_\perp\, d\sigma = 0$. (*) Indeed, $U_\perp = \mathrm{Re}\sum e^{iT(l+\frac{d-1}{2})}\hat{F}_\perp(l,m)Y_{l,m}$

and $F_\perp$ comes from $f_\perp$ which satisfies $\langle f_\perp | f_*\rangle = 0$.

Thus, $\langle F_\perp | 1\rangle = 0$ and this implies $\hat{F}_\perp(0,m) = 0$.

We infer (*). $\square$