Volumes spanned by k-point configurations in $\mathbb{R}^d$

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(joint work with Belmiro Galo)

Point Distribution Webinar
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The Falconer distance problem

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- Falconer (1986) proved that the threshold $\dim E > \frac{d+1}{2}$ was sufficient, and that no threshold below $\frac{d}{2}$ is sufficient. The conjectured best threshold is $\frac{d}{2}$.

Results on the Falconer problem

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- The best current results are

$$\begin{cases} 5/4, & d = 2 (Guth, Iosevich, Ou, Wang) \\ 9/5, & d = 3 (Du, Guth, Iosevich, Ou, Wang, Zhang) \\ \frac{d}{2} + \frac{1}{4}, & d \geq 4, d \text{ even } (Du, Iosevich, Ou, Wang, Zhang) \\ \frac{d}{2} + \frac{1}{4} + \frac{1}{4(d-1)}, & d \geq 4, d \text{ odd } (Du, Zhang) \end{cases}$$

Strategy for Falconer problem

- If E is compact, for any $s < \dim E$ there is a probability measure μ supported on E such that

$$\mu(B_r(x)) \lesssim r^s$$

and

$$I_s(\mu) := \int \int |x-y|^{-s} \, d\mu(x) \, d\mu(y) < \infty.$$

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- The measure μ is called a Frostman probability measure with exponent s.

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- ν is a probability measure supported on $\Delta(E)$, so to prove $\Delta(E)$ has positive Lebesgue measure it suffices to show ν is absolutely continuous.

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- For $A \subset \mathbb{R}$, we have

$$\int_A \nu^\varepsilon(t)\, dt \lesssim |A|^{1/2} \|\nu^\varepsilon\|_{L^2}.$$

- The left hand side has limit $\nu(A)$, so it suffices to prove a bound on $\|\nu^\varepsilon\|_{L^2}$ which is independent of ε.

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- Given $\Phi : (\mathbb{R}^d)^N \to \mathbb{R}^M$, define ν by

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$$\int f(t)\, d\nu(t) = \int f(\Phi(x^1, ..., x^N))\, d\mu(x^1) \cdots d\mu(x^N).$$

- If $\|\nu^\varepsilon\|_{L^2}$ is bounded independent of ε, then $\{\Phi(x^1, ..., x^N) : x^i \in E\}$ has positive measure.

Congruence of point configurations

- A $(k+1)$-point configuration in $\mathbb{R}^d$ is simply an element of $(\mathbb{R}^d)^{k+1}$, i.e., a $k+1$ tuple $x = (x^1, ..., x^{k+1})$ where each $x^i = (x^i_1, ..., x^i_d)$ is a vector in $\mathbb{R}^d$.

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- We say $(k+1)$-point configurations x and y are congruent, and write $x \sim y$, if there exists $\theta \in O(\mathbb{R}^d), z \in \mathbb{R}^d$ such that for all $i = 1, ..., k+1$ we have $y^i = \theta x^i + z$ (briefly, $y = \theta x + z$).

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- Given $E \subset \mathbb{R}^d$, let $\Delta_k(E)$ denote the set of congruence classes determined by E.

- We may identify $\Delta(E)$ with $\Delta_1(E)$.

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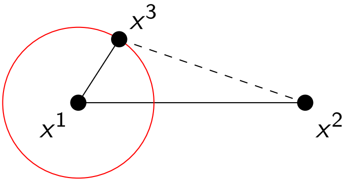
- The choice of measure depends on whether $k \leq d$ or $k > d$.

- When $k \leq d$, each of the pairwise distances may be chosen independently. We may therefore identify $\Delta_k(E)$ with a subset of $\mathbb{R}^{\binom{k+1}{2}}$, equipped with $\binom{k+1}{2}$-dimensional Lebesgue measure.

Example: The case $k = d = 2$



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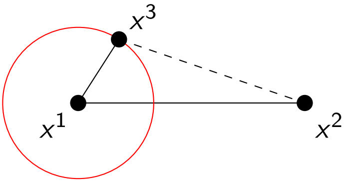

- Let $x = (x^1, x^2, x^3)$ be a 3-point configuration in $\mathbb{R}^2$.

- If we fix $|x^1 - x^2| = a$ and $|x^1 - x^3| = b$, the last distance $|x^2 - x^3|$ could take any value between $|a - b|$ and $a + b$.

Configuration congruence results in the $k \leq d$ case

Theorem (Greenleaf-Iosevich-Liu-Palsson, 2015)

Let $k \leq d$, and let $E \subset \mathbb{R}^d$ be a compact set. If

$$\dim E > d - \frac{d-1}{k+1},$$

then $\Delta_k(E)$ has positive $\binom{k+1}{2}$-dimensional Lebesgue measure.

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- In the case $k = 1$ this coincides with Falconer's $\frac{d+1}{2}$ threshold.

Overdetermined configurations

- When $k > d$, the system of equations

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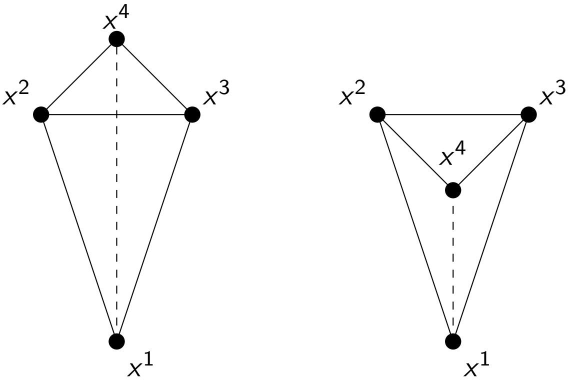
- In this case we may still identify $\Delta_k(E)$ with $\binom{k+1}{2}$-tuples of pairwise distances, but the resulting subset of $\mathbb{R}^{\binom{k+1}{2}}$ has measure zero.

Example: The case $k = 3$, $d = 2$



- With 4 points, if we fix 5 of the pairwise distances there are only 2 choices for the last distance.

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- Say that a configuration x is non-degenerate if $x^1, \ldots, x^{d+1}$ are affinely independent.

- Two non-degenerate configurations x, y are congruent if and only if there exists $\theta \in O(\mathbb{R}^d), z \in \mathbb{R}^d$ such that $y = \theta x + z$.

- The non-degenerate congruence classes can be identified with a space of dimension m, where

$$m = d(k+1) - \binom{d+1}{2}$$

Overdetermined congruence result

Theorem (Chatzikonstantinou-Iosevich-Mkrtchyan-Pakianathan, 2017)

Let $d \geq 2$ and $k \geq 1$, and let $m = d(k+1) - \binom{d+1}{2}$. Let $E \subset \mathbb{R}^d$ be compact. If $\dim E > d - \frac{1}{k+1}$, then $\Delta_k(E)$ has positive m-dimensional measure.

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- This approach generalizes to other overdetermined configuration problems if the relevant geometric features can be characterized in terms of a group action.

Volumes

Theorem (Greenleaf-Iosevich-Taylor, 2020)

Given $E \subset \mathbb{R}^d$, *define*

$$\mathcal{V}(E) = \{\det(x^1, ..., x^d) : x^i \in E\}$$

If $E \subset \mathbb{R}^d$ *is compact and* $\dim E > d - 1 + \frac{1}{d}$, *then* $\mathcal{V}(E)$ *has non-empty interior.*

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- If $\dim E \leq d - 1$ then E may be contained in a hyperplane and determine no non-trivial volumes.
- It follows the threshold $d - 1 + \frac{1}{d}$ cannot be improved by more than $\frac{1}{d}$.

Volume types of point configurations

- Let x be a k-point configuration in $\mathbb{R}^d$, i.e.,

$$x = (x^1, ..., x^k)$$

  for vectors

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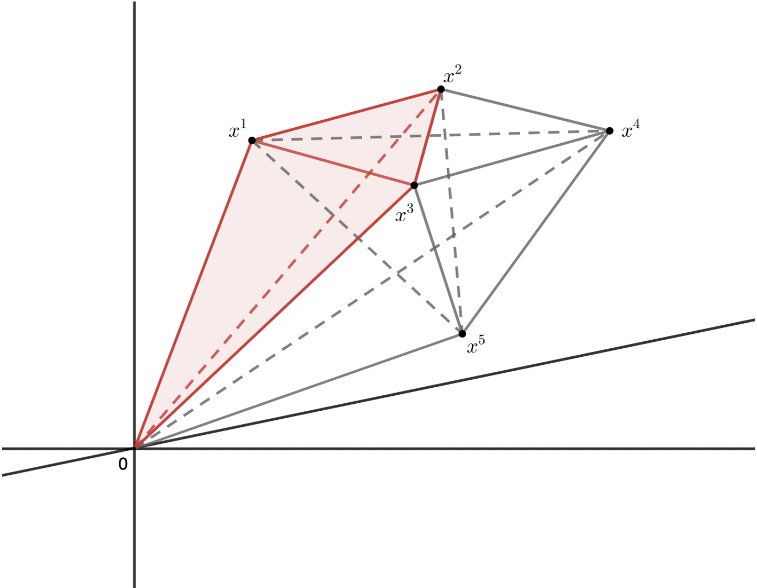
- The **volume type** of x is the vector

$$\{\det(x^{i_1}, ..., x^{i_d})\}_{1 \leq i_1 < \cdots < i_d \leq k} \in \mathbb{R}^{\binom{k}{d}}.$$

A 5-point configuration in $\mathbb{R}^3$



Volume types of configurations

- Given $k \geq d$ and $E \subset \mathbb{R}^d$, let $\mathcal{V}_{k,d}(E)$ denote the set of volume types determined by configurations of points in E. Let $\mathcal{V}_{k,d} = \mathcal{V}_{k,d}(\mathbb{R}^d)$.

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- Let $\Phi_{k,d} : (\mathbb{R}^d)^k \to \mathcal{V}_{k,d}$ be the map taking configurations to their volume types.

- If $g \in \mathrm{SL}_d(\mathbb{R})$, it is clear that $\Phi_{k,d}(gx) = \Phi_{k,d}(x)$.

Volume types and the action of $\mathrm{SL}_d(\mathbb{R})$

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Volume types and the action of $\mathrm{SL}_d(\mathbb{R})$

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- Define
$$g = (y^1, ..., y^d)(x^1, ..., x^d)^{-1}.$$
- Then $g \in \mathrm{SL}_d(\mathbb{R})$, and $gx^i = y^i$ for $i = 1, 2, ..., d$.

Volume types and group actions

- For $i > d$, write

$$x^i = \sum_{j=1}^{d} a_{i,j} x^j, \qquad y^i = \sum_{j=1}^{d} b_{i,j} y^j.$$

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- Easy to prove $a_{i,j} = b_{i,j}$, so $gx^i = y^i$ for all i.

- For every non-degenerate x, there exists $\widetilde{x}$ of the form

$$\widetilde{x} = (e^1, \ldots, e^{d-1}, te^d, z^{d+1}, \ldots, z^k)$$

with $\Phi_{k,d}(\widetilde{x}) = \Phi_{k,d}(x)$.

Theorem

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- With this identification, our result is as follows.

Theorem (Galo-M., 2021)

Let $k \geq d \geq 2$, let $m = d(k-d)+1$, and let $E \subset \mathbb{R}^d$ be compact. If $\dim E > d - \frac{d-1}{2k-d}$, then $\mathcal{L}_m(\mathcal{V}_{k,d}(E)) > 0$.

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- If $k = d$, then our threshold is $d - 1 + \frac{1}{d}$, which is the threshold in the Greenleaf-Iosevich-Taylor result.

Setup

- If E is compact, for any $s < \dim E$ there is a probability measure μ supported on E such that

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- Let $\nu_{k,t}^{\varepsilon}$ be the convolution of $\nu_{k,t}$ with an approximate identity. Our goal is to prove L^2 bounds on $\nu_{k,t}^{\varepsilon}$, independent of ε.

The $k = d$ case

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- Let ψ be a Schwartz function supported in the range $\frac{1}{2} \leq |\xi| \leq 4$ and constantly equal to 1 in the range $1 \leq |\xi| \leq 2$, and let $\widehat{\mu_j}(\xi) = \psi(2^{-j}\xi)\widehat{\mu}(\xi)$ be the corresponding Littlewood-Paley projection.

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- The above integral is

$$\varepsilon^{-1} \sum_{j_1 > \cdots > j_d > 0} \int_{|\det(x^1,\dots,x^d)-t|<\varepsilon} \mu_{j_1}(x^1) \cdots \mu_{j_d}(x^d)\, dx$$

The $k = d$ case

- Define a generalized Radon transform by

$$\mathcal{R}_t f(x^1, \cdots x^{d-1}) = \int_{\substack{\det(x^1,\cdots,x^d)=t \\ |x^1|,\ldots,|x^d|\leq 1}} f(x^d)\, d\sigma_{t,x^1,\cdots,x^{d-1}}(x^d),$$

where $\sigma_{t,x^1,\cdots,x^{d-1}}$ is the surface measure.

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- The Greenleaf-Iosevich-Taylor result is obtained from this by studying the mapping properties of generalized Radon transforms.

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- This, together with Plancherel, gives bounds on the L^2 inner product

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The sum is finite when $s > d - 1 + \frac{1}{d}$.

Reducing to the $k = d$ case

- For general $k \geq d$, we have

$$\nu_{k,d}^{\varepsilon}(t) \approx \varepsilon^{-m} \int_{|\Phi_{k,d}(x)-t|<\varepsilon} d\mu^k(x).$$

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- Applying the bound $\|\mu_j\|_{L^\infty} \leq 2^{j(d-s)}$ to the last $k-d$ terms, this is

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- The sum is finite when $s > d - \frac{d-1}{2k-d}$.

Sharpness

Theorem (Galo-M.,2021)

Let $k \geq d \geq 2$. For any

$$s < d - \frac{d^2(d-1)}{d(k-1)+1},$$

there exists compact $E \subset \mathbb{R}^2$ such that $\dim E = s$ and $\mathcal{V}_{k,d}(E)$ has measure zero.

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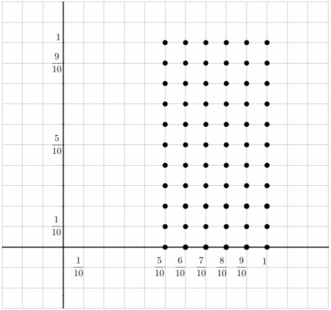
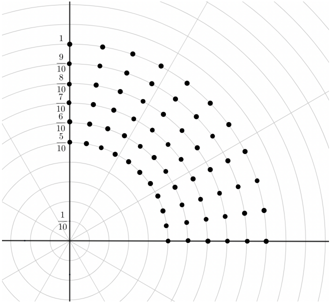
- This approximates a set of dimension s in $\mathbb{R}^d$.
- Map the square lattice to a spherical lattice.

Sharpness





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- The thickened set has an volume type set of measure $\approx q^{d(k-1)+1}\left(q^{-d/s}\right)^{d(k-d)+1}$.
- If $s < d - \frac{d^2(d-1)}{d(k-1)+1}$, this tends to zero as $q \to \infty$.

Distance chains

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- The following result applies when G is a chain.

Theorem (Bennett-Iosevich-Taylor, 2015)

Let $d, k \geq 2$, and let $E \subset \mathbb{R}^d$ be compact. If $\dim E > \frac{d+1}{2}$, the set

$$\{(|x^1 - x^2|, ..., |x^{k-1} - x^k|) : x^i \in E\}$$

has non-empty interior.

Distance trees

- This result was later generalized from chains to trees.

Theorem (Iosevich-Taylor, 2019)

Let $d, k \geq 2$ and let $E \subset \mathbb{R}^d$ be compact. Let T be a tree on the vertices $\{1, ..., k\}$ with edge set $\mathcal{E}$. If $\dim E > \frac{d+1}{2}$, the set

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- For both chains and trees, the threshold does not depend on k.

Volume chains

- Our second result is an analogue for hypergraph chains of volumes.

Theorem (Galo-M., 2021)

Let $k, d \geq 2$ and let $E \subset \mathbb{R}^d$ be compact. If $\dim E > d - 1 + \frac{1}{d}$, then

$$\{\{\det(x^j, x^{j+1}, \cdots, x^{j+d-1})\}_{1 \leq j \leq k+1-d} : x^1, ..., x^k \in E\}.$$

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The $k = d + 1$ case

- Suppose $k = d + 1$. The quantity we want to bound is

$$\varepsilon^{-2} \int_{\substack{|\det(x^1,\ldots,x^d)-t|<\varepsilon \\ |\det(x^2,\ldots,x^{d+1})-t'|<\varepsilon}} d\mu^{d+1}(x)$$

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- $$\approx \varepsilon^{-1} \sum_j \int_{|\det(x^1,\ldots,x^d)-t|<\varepsilon} \mathcal{R}_{t'} \mu_j(x^2, \ldots, x^d) d\mu^d(x).$$

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- One can handle arbitrary $k \geq d$ by iterating this process.

The End

Thanks for listening!