Maximizing angle sums between Euclidean lines

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A family of generalized Fejes Tóth problems

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Find a globally stable configuration of k charges on projective sphere $\mathbf{RP}^n$, which repel each other pairwise with a force independent of distance.

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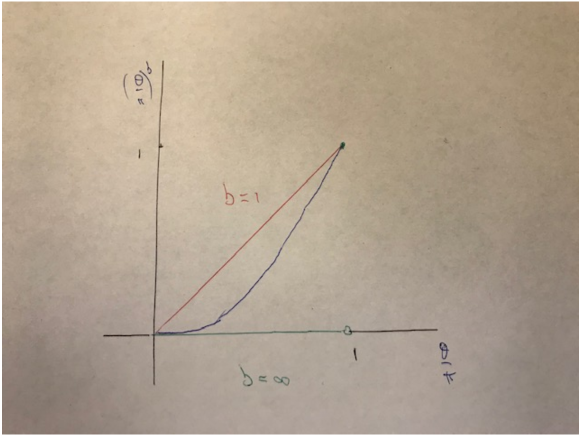
- obvious for $k \leq n+1$; also resolved affirmatively for all k with $n = 1$ but remains open otherwise

- for $n = 1$ and $k > 2$ there are many inequivalent maximimizers as well (some of which accumulate to the uniform distribution)

To make progress, let the force increase with a power $b-1>0$ of the distance, so we maximze $\sum(\frac{\theta}{\pi})^b$ instead



Identifying $\pm x \in \mathbf{S}^n$ yields $\mathbf{RP}^n = \frac{2}{\pi}\mathbf{S}^n/\{+,-\}$ scaled to diameter 1. Let

$$d_{\mathbf{RP}^n}(x,y) = \frac{2}{\pi}\min\{d_{\mathbf{S}^n}(x,y), \pi - d_{\mathbf{S}^n}(x,y)\}$$

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Fejes Tóth's problem becomes the case $b = 1$ of

$$\max_{\mu \in \mathcal{P}_k^{=}(\mathbf{RP}^n)} E_b(\mu)$$

where

$$E_b(\mu) = \frac{1}{2}\iint d_{\mathbf{RP}^n}(x,y)^b d\mu(x) d\mu(y)$$

F.T. Conjecture: standard basis $\{\hat{e}_1, \ldots, \hat{e}_{d+1}\}$ (i.e. maximal projective simplex) $\bar{\mu}_k = \frac{1}{k}\sum_{i=1}^{k}\delta_{\hat{e}_{(k \mod d+1)}}$ achieves maximum for all $b \in [1,\infty]$.

Theorem (Discrete threshold for simplex maxima)

(a) For $k > n+1$, there exists $b_{\Delta^n}(k) \in [1, \infty)$ such that $\bar{\mu}_k$ maximizes $E_b(\mu)$ on $\mathcal{P}_k^{=}(\mathbf{RP}^n)$ if and only if $b \geq b_{\Delta^n}(k)$.
(b) $\bar{\mu}_k$ maximizes uniquely up to rotations if and only if $b > b_{\Delta^n}(k)$.

RMK: Fejes Tóth conjecture $\Longleftrightarrow$ $b_{\Delta^n}(k) = 1$ for all $k > n+1$

Note $2E_b(\bar{\mu}) = \frac{n}{n+1}$ for $\bar{\mu} = \bar{\mu}_k$ when $n+1$ divides k.

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Theorem (Continuum threshold for simplex maxima)

If $n+1$ divides k then there exists $b_{\Delta^n} \in [1, \infty)$ such that
(c) $\bar{\mu} = \mu_k$ maximizes E_b over the full set $\mathcal{P}(\mathbf{RP}^n)$ iff $b \geq b_{\Delta^n}$,
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(e) $b_{\Delta^n} < 2$

RMK: (e) improves $b_{\Delta^n} \leq 2$ (Bilyk, Glazyrin, Matzke, Park, Vlasiuk)

$b = 2$ is the threshold for *mild repulsion* in attractive-repulsive models

ASIDE: a different but related energy optimization

Balagué, Carrillo, Laurent, Raoul '13; Attractive-repulsive potentials
If

$$\mu \in \arg \min_{\mu \in \mathcal{P}(\mathbf{R}^n)} E_{a,b}(\mu)$$

for $b \in (2-n, a)$ where

$$E_{a,b}(\mu) := \iint_{\mathbf{R}^n \times \mathbf{R}^n} (\frac{|x|^a}{a} - \frac{|x|^b}{b}) d\mu(x) d\mu(y)$$

then $spt\,\mu$ has Hausdorff dimension $\geq 2-b$

e.g. $a = 2$

Easy proof (apart from 'only if'):

- $0 \leq d_{\mathbf{RP}^n}(x, y) \leq 1$ with equality iff $x = y$ or $x \perp y$
- $d_{\mathbf{RP}^n}(x, y)^b \geq d_{\mathbf{RP}^n}(x, y)^{b+\epsilon}$ with the same conditions for equality
- $(spt\, \hat{\mu}_k) = \{\hat{e}_1, \ldots, \hat{e}_{n+1}\}^2$ lies in the equality set
- if $\hat{\mu}_k$ maximizes E_b for some b, it also maximized for all $b + \epsilon > b$
- and every measure not supported in the equality set does strictly worse

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- nonunique $n = b = 1 = \frac{k}{3}$ maximizer implies $b_{\Delta^n}(k) \geq 1$ if $k > n+1$
- we argued $b_{\Delta^n}(k) < \infty$ in an earlier work, simplified by Bilyk et al, who also observed $k < \infty$ follows as a consequence of Turan's (1941) theorem, which identifies the graph $T(k, n+1) = K_{\lceil \frac{k}{n+1} \rceil, \ldots, \lceil \frac{k}{n+1} \rceil, \lfloor \frac{k}{n+1} \rfloor, \ldots, \lfloor \frac{k}{n+1} \rfloor}$ free of $(n+2)$-cliques that has the maximal number of edges

- 'only if' asserts $\mathrm{argmax}\, E_\infty \subset \mathrm{argmax}\, E_{b_{\Delta^n}(k)}$ strictly

‘only if’ follows from a local optimality result $\forall b > 1$:

OPTIMAL TRANSPORT:
L^p-Kantorovich-Rubinstein-Wasserstein distance d_p on $\mu, \nu \in \mathcal{P}(\mathbf{RP}^n)$

$$d_p(\mu,\nu) := \inf_{X\sim\mu, Y\sim\nu} \|d_{\mathbf{RP}^n}(X,Y)\|_{L^p} \qquad p \in [1,\infty]$$

$$= \min_{0\le\gamma\in\Gamma(\mu,\nu)} \left(\iint_{\mathbf{RP}^n\times\mathbf{RP}^n} d_{\mathbf{RP}^n}(x,y)^p d\gamma(x,y) \right)^{1/p} \qquad p \in [1,\infty)$$

- $p < \infty$: metrizes weak-$*$ convergence of measures
- $d_\infty = \lim_{p\to\infty} d_p$ metrizes a much finer topology
- in this finer topology, $E_b(\mu)$ can have more local maxima
- c.f. McCann (PhD 1994, HJM 2006) stable binary stars

Weighted maximal projective simplices

For $0 \le m_1 \le m_2 \le \cdots \le m_{n+1}$ with $\sum_{i=1}^{n+1} m_i = 1$ set $\vec{m} = (m_1, \ldots, m_{n+1})$

$$\mu_{\vec{m}} = \sum_{i=1}^{n+1} m_i \delta_{\hat{e}_i}$$

Theorem (Weighted simplices are strict d_∞-locally maxima $\forall b > 1$)

Given $\epsilon > 0$ and $n \in \mathbf{N}$ there exists $r = r_n(\epsilon)$ such that if $b \ge 1 + \epsilon$, $m_1 \ge \epsilon$ and $\mu \in \mathcal{P}(\mathbf{RP}^n)$ satisfy $d_\infty(\mu, \mu_{\vec{m}}) < r$ then $E_b(\mu) \le E_b(\mu_{\vec{m}})$ and the inequality is strict unless μ is a rotate of $\mu_{\vec{m}}$

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- crucial for proving $\operatorname{argmax} E_\infty \subset \operatorname{argmax} E_{b_{\Delta^n}(k)}$ strictly (i.e. 'only if')
- e.g. $\mu_\epsilon \in \operatorname*{argmax}_{\mathcal{P}_k^=(\mathbf{RP}^n)} E_{b_{\Delta^n}(k)-\epsilon}$ d_2-accumulates to $\mu_0 \in \operatorname*{argmax}_{\mathcal{P}_k^=(\mathbf{RP}^n)} E_{b_{\Delta^n}(k)}$
- on the discrete measures $\mathcal{P}_k^=(\mathbf{RP}^n)$, d_∞ gives the same topology as d_2

• On the full space $\mathcal{P}(\mathbf{R}^n)$, the argument is more subtle: if $d_2(\mu_\epsilon, \bar{\mu}) \to 0$ the Euler-Lagrange equation satisfied by μ_ϵ implies $d_\infty(\mu_\epsilon, \mu_{\vec{m}}) \to 0$ for *spt* $\mu_{\vec{m}}$ an approximately maximal simplex with approximately uniform weights $\vec{m} = (m_1, \ldots, m_{n+1})$; energetic maximality of μ_ϵ implies both approximations are exact

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• The d_∞-local maximizer theorem relies on the following estimate, which follows approximately from control of the ℓ_2 norm on $\mathbf{R}^n$ by the ℓ_1 norm

Lemma

Given $C \in (0,1)$ there exists $r = r_n(C) > 0$ such that $d_\infty(\nu_i, \delta_{\hat{e}_i}) < r$ for all $i = 1, \ldots, n$ implies $\exists \bar{x} = \bar{x}(\nu_1, \ldots, \nu_n)$ such that

$$\sum_{i=1}^{n} \int (1 - d_{\mathbf{RP}^n}(x, y)) d\nu_i(y) \geq C d_{\mathbf{RP}^n}(x, \bar{x}) \quad \text{whenever } d_{\mathbf{RP}^n}(x, x_{n+1}) < r$$

For $b > 1$, if $d_\infty(\mu, \mu_{\vec{m}}) < r$, this can be used to show we gain more energy than we lose by collapsing any mass of μ near $\hat{e}_{n+1}$ to $\bar{x}$

Further evidence for Fejes Tóth's conjecture?

Corollary (Discontinuous bifurcation unless $b_{\Delta^n} = 1$)

Fix $n \in \mathbf{N}$. No curve $(\mu_b)_{b>0}$ of optimizers

$$\mu_b \in \operatorname*{argmax}_{\mathcal{P}(\mathbf{RP}^n)} E_b$$

can be d_∞-continuous at $b = b_{\Delta^n}$ except possibly if $b_{\Delta^n} = 1$.

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Corollary (On connectedness of the threshold set of optimizers)

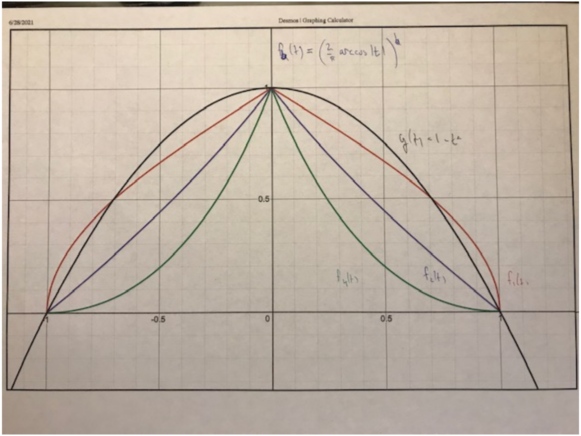
Fix $n \in \mathbf{N}$. If the set of threshold optimizers

$$\operatorname*{argmax}_{\mathcal{P}(\mathbf{RP}^n)} E_{b_{\Delta^n}}$$

forms a d_∞-connected subset of $\mathcal{P}(\mathbf{RP}^n)$, then $b_{\Delta^n} = 1$.

Bilyk-Glazyrin-Matzke-Park-Vlasiuk bound: $b_{\Delta^n} \leq 2$

Set $f_b(t) := (\frac{2}{\pi} \arccos(t))^b$ and $g(t) := 1 - t^2$



Proof: $f_b(t) := (\frac{2}{\pi} \arccos(t))^b$ yields

$$E_b(\mu) = \iint_{\mathbf{RP}^n \times \mathbf{RP}^n} f_b(x \cdot y) d\mu(x) d\mu(y) =: F^{f_b}(\mu)$$

For $b \geq 2$ we have $f_b(t) \leq g(t) := 1 - t^2$ with equality iff $t \in \{-1, 0, 1\}$. Thus

$$E_2(\mu) \leq F^g(\mu) \leq F^g(\sigma) = F^g(\bar{\mu}) = E_2(\bar{\mu})$$

where $\sigma =$ *normalized surface area* and $\bar{\mu} = \frac{1}{n+1} \sum \delta_{\hat{e}_i}$. The first inequality is strict unless $\mu = \mu_{\vec{m}}$ for some $\vec{m} = (m_1, \ldots m_{n+1})$.

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$$\iint_{\mathbf{S}^n \times \mathbf{S}^n} (x \cdot y)^2 d\mu(x) d\mu(y) = Tr[I(\mu)^2] \geq \frac{1}{n+1}(Tr\, I(\mu))^2 = \frac{1}{n+1}$$

for the moment of inertia tensor

$$I(\mu) = (I_{ij}(\mu))_{i,j=1}^{n+1} \qquad I_{ij}(\mu) = \int_{\mathbf{R}^n} x_i x_j d\mu(x).$$

QED

Thank you!