A sequence of well conditioned polynomials

Fátima Lizarte

Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria

Point Distributions Webinar

Structure

Problem: Find explicitly a sequence of univariate polynomials P_N of degree N with condition number bounded above by N. It is posed by Shub and Smale in 1993.

C. Beltrán and F. Lizarte, *On the minimum value of the condition number of polynomials*, 2021, arXiv: 2012.05138.

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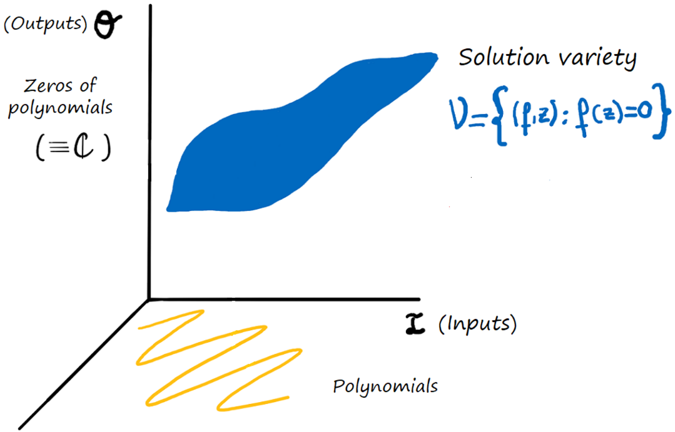
Contents:

- What is the condition number of polynomials?
- M. Shub and S. Smale. Origin.
- Previous knowledge until this work.
- Main result.
- Comments on the proof of the main result.
- References.


(Outputs) 𝔒
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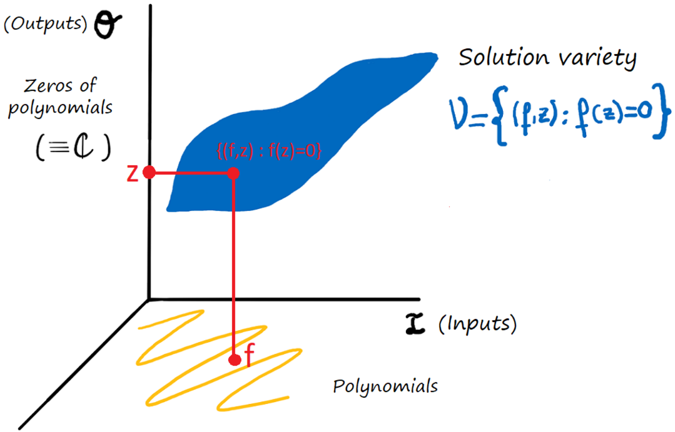
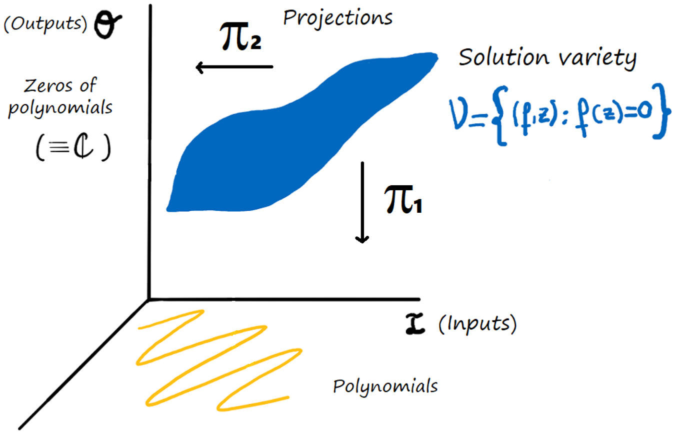
(≡ℂ)
Solution variety
V={(f,z): f(z)=0}
𝔍 (Inputs)
Polynomials

Condition number of polynomials

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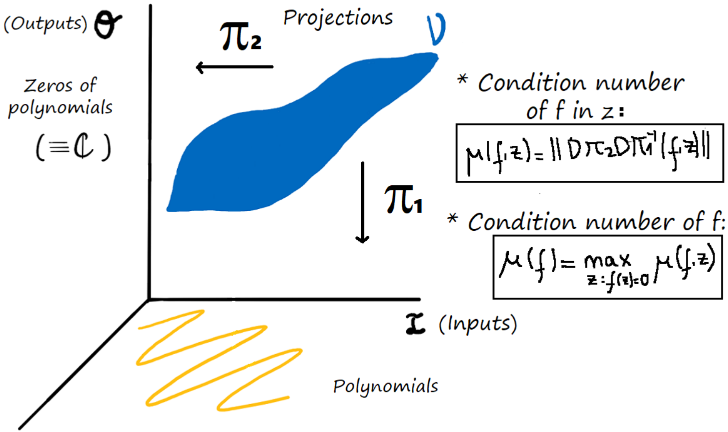
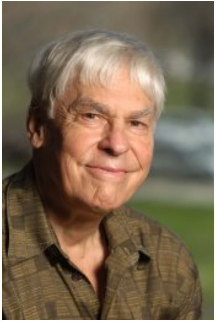


M. Shub & S. Smale

Michael Shub (1943 -)
American mathematician
Dynamical Systems
Complexity theory



Stephen Smale (1930 -)
American mathematician
Dynamical Systems
Topology
Theories of computation
Mathematical economics
Fields Medal

https://mariposa-arts.net/GalleryMain.asp?GalleryID=175191&AKey=TJMS9E5S

https://math.berkeley.edu/ smale/crystals.html

Smale's problems

They are a list of 18 challenging problems for the twenty-first century proposed by Stephen Smale and chosen with these criteria:

- Simple statement.

- Personal acquaintance with the problem.

- A belief that the question, its solution, partial results or even attempts at its solution are likely to have great importance for mathematics and its development in the 21st century.

S. Smale, *Mathematical problems for the next century, Mathematics: frontiers and perspectives*, 2000, pp. 271–294.

Smale's 17th problem

It is about the search for an efficient algorithm (polynomial time) to compute zeros of systems of polynomial equations.

Homotopy method: based on solving a system from another whose solution is known.

Shub and Smale proved that this method can be done in a totally rigorous way once the initial pair has been correctly selected.

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⋆ Solution:

Beltrán and Pardo: probabilistic algorithm with polynomial time complexity.
Bürgisser and Cucker: deterministic algorithm with quasi-polynomial time complexity.
Lairez: deterministic algorithm with polynomial time complexity.

Previous knowledge

Shub and Smale proved that:

- If $f(z) = \sum_{i=0}^{N} a_i z^i$ with $a_i \sim \mathcal{N}_{\mathbb{C}}(0, \binom{N}{i})$ is a random polynomial of degree $N \Rightarrow$ the probability that $\mu(f) \leqslant N$ is $\geqslant 1/2$.

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★ The relaxed version of the Shub and Smale problem: finding explicitly a family of polynomials P_N of degree N such that $\mu(P_N) \leqslant N^{c}$.

Previous knowledge. Relation to potential theory

The logarithmic energy of $\omega_N = (\widehat{z}_1, \ldots, \widehat{z}_N) \in \mathbb{S}^2 \subseteq \mathbb{R}^3$, a collection of N points:

$$\mathcal{E}(\widehat{z}_1, \ldots, \widehat{z}_N) = \sum_{i \neq j} \ln \frac{1}{|\widehat{z}_i - \widehat{z}_j|}$$

Notation: $\mathcal{E}_N = \min_{\widehat{z}_1, \ldots, \widehat{z}_N \in \mathbb{S}^2} \mathcal{E}(\widehat{z}_1, \ldots, \widehat{z}_N)$

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- If N spherical points such that

$$\mathcal{E}(\widehat{z}_1, \ldots, \widehat{z}_N) \leqslant \mathcal{E}_N + c \ln N$$

are known, with c a constant, then one can construct a solution to the relaxed of Shub and Smale problem.

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How?

Just taking the polynomial whose zeros are obtained from the stereographic projection of the N known spherical points whose condition number is at most $N^{1+c/2}$.

Smale's 7th problem

To find $\widehat{z}_1, \ldots, \widehat{z}_N \in \mathbb{S}^2$ such that

$$\boxed{\mathcal{E}(\widehat{z}_1, \ldots, \widehat{z}_N) \leqslant \mathcal{E}_N + c \ln N}$$

$$\mathcal{E}_N = \kappa N^2 - \frac{1}{2} N \ln N + C_{\log} N + o(N)$$

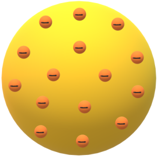
- Continuous energy:

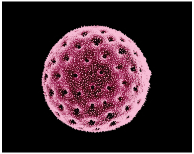
$$\kappa = \int_{x,y\in\mathbb{S}^2} \ln \frac{1}{|x-y|} d\sigma(x) d\sigma(y) = \frac{1}{2} - \ln 2 < 0$$

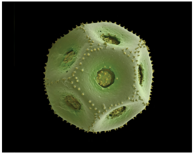
- Constant $C_{\log}$:

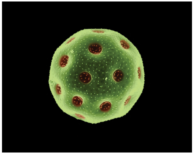
$$-0.0569\ldots \leqslant C_{\log} \leqslant 2\ln 2 + \frac{1}{2} \ln \frac{2}{3} + 3 \ln \frac{\sqrt{\pi}}{\Gamma(1/3)}$$

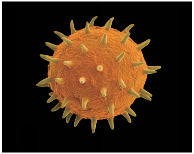
Smale's 7th problem

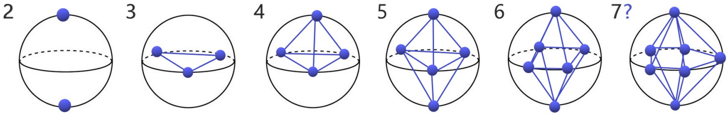












Kesseler, R. and Harkey M., (2011). *Polen: la sexualidad oculta de las flores*. Turner.

Solution

It has been recently resolved in:

C. Beltrán, U. Etayo, J. Marzo and J. Ortega-Cerdà, *A sequence of polynomials with optimal condition number*, J. Am. Math. Soc., 2020.
DOI: 10.1090/jams/956

through a complex process:

- By a closed formula for large enough and unknown N.
- By a search algorithm for the rest of the cases.

⋆ Optimal value of the condition number: $\mathcal{O}(\sqrt{N})$.

Main result

Theorem

Let $N = 4M^2$, with $M \geqslant 1$ a positive integer. Define

$$r_j = 4j, \quad h_j = 1 - \frac{4j^2}{N},$$

for $1 \leqslant j \leqslant M$ and consider the polynomial of degree N as follow

$$\boxed{P_N(z) = (z^{r_M} - 1) \prod_{j=1}^{M-1} (z^{r_j} - \rho(h_j)^{r_j})(z^{r_j} - \rho(h_j)^{-r_j})}$$

where $\rho(x) = \sqrt{\frac{1+x}{1-x}}$. Then $\mu_{\text{norm}}(P_N) \leqslant \min(N, (19/2)\sqrt{N+1})$.

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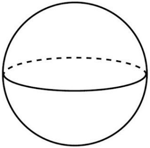
M	N	Polynomial
1	4	$z^4 - 1$
2	16	$(z^8 - 1)(z^4 - 49)(z^4 - 1/49)$
3	36	$(z^{12} - 1)(z^8 - 2401/16)(z^8 - 16/2401)(z^4 - 289)(z^4 - 1/289)$

1. Stereographic projection



Main result. Ingredients for the proof

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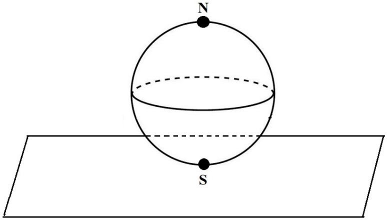

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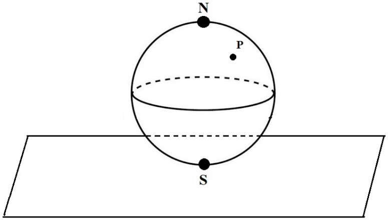

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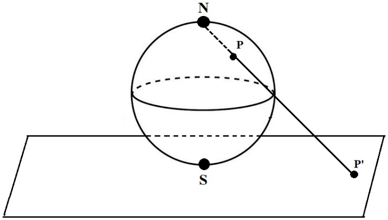

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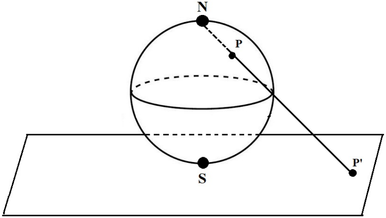


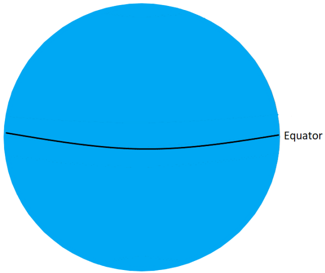
Let (x, y, z) be Cartesian coordinates on $\mathbb{S}^2$ and (X, Y) on the plane:

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$
$$(x, y, z) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{X^2+Y^2-1}{1+X^2+Y^2} \right)$$

Main result. Ingredients for the proof

2. Construction of a set of N spherical points $\mathcal{P}_N$

- $\mathcal{P}_N = \{p_1, \ldots, p_N\} \subseteq \mathbb{S}^2$
- Total number of points: $N = 4M^2$
- Symmetry with respect to the equator



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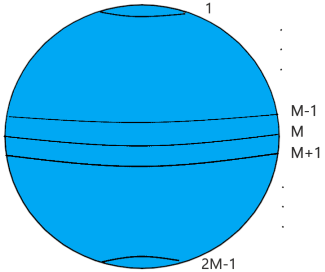

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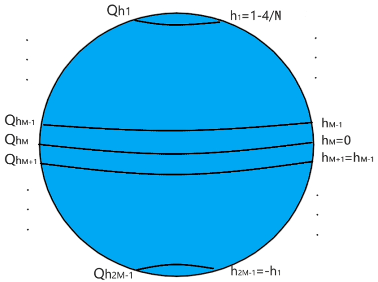

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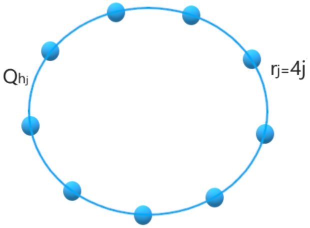

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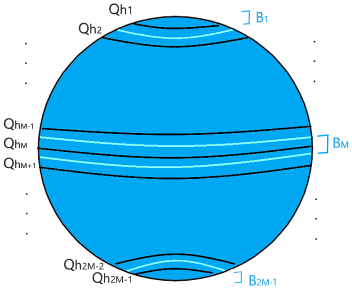

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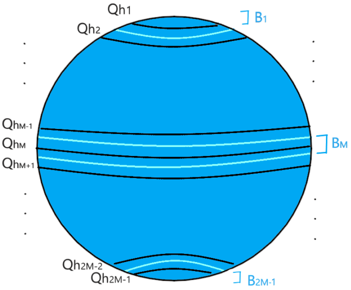

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- Band: $B_j = \left\{(x, y, z) \in \mathbb{S}^2 \ : \ h_j - \frac{r_j}{N} \leqslant z \leqslant h_j + \frac{r_j}{N}\right\}$
- By symmetry:
  $$r_{M+j} = r_{M-j}, \quad h_{M+j} = -h_{M-j}, \quad 1 \leqslant j \leqslant M - 1$$

2. Construction of a set of N spherical points $\mathcal{P}_N$

The points $\mathcal{P}_N$ are the set of roots of unity in the circle defined by the parallels. That is, for $i = 0, \ldots, j-1$ and $1 \leqslant j \leqslant M$,

$$p_{i,r_j} = \left(\sqrt{1-h_j^2} \cos\left(\frac{2\pi i}{r_j}\right), \sqrt{1-h_j^2} \sin\left(\frac{2\pi i}{r_j}\right), h_j \right)$$

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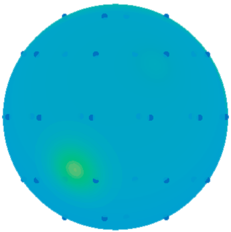
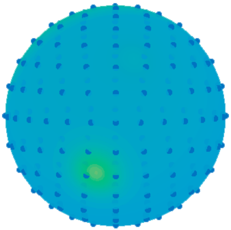
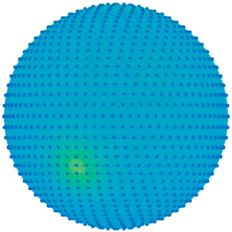


$\mathcal{P}_{36}$ (For $M = 3$)



$\mathcal{P}_{196}$ (For $M = 7$)



$\mathcal{P}_{1600}$ (For $M = 20$)

3. Formula for condition number

Let $P(z) = \prod_{i=1}^{N}(z - z_i)$ be a polynomial and denote by $\widehat{z}_i \in \mathbb{S}^2$ the point in $\mathbb{S}^2$ obtained from the stereographic projection of z_i. Then

$$\mu(P) = \frac{1}{2}\sqrt{N(N+1)} \max_{1 \leqslant i \leqslant N} \frac{\left(\int_{\mathbb{S}^2} \prod_{j=1}^{N} |p - \widehat{z}_j|^2 d\sigma(p)\right)^{1/2}}{\prod_{i \neq j} |\widehat{z}_i - \widehat{z}_j|}$$

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We get a upper bound from the previous formula! For $p \in B_\ell$ and $M \geqslant 5$:

$$\prod_{i=1}^{N} |p - p_i|^2 \leqslant 4e^{3/2} e^{-2\kappa N} \left(\frac{M}{\ell}\right)^{2/3} \left(\frac{e}{2}\right)^{4/\ell}$$

$$\prod_{i \neq j} |p_i - p_j| \geqslant \sqrt{2N} e^{-9.8} e^{-\kappa N}$$

Main result. Ingredients for the proof

For $M \leqslant 4$?

Our proof is computer assisted from the expression

$$\mu(f,z) = \frac{N^{1/2}(1+|z|^2)^{\frac{N-2}{2}}}{|f'(z)|}\|f\|$$

with

$$f(z) = \sum_{i=0}^{N} a_i z^i$$

and Bombieri-Weyl norm

$$\|f\| = \left(\sum_{i=0}^{N} \binom{N}{i}^{-1} |a_i|^2\right)^{1/2}$$

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Thank you!