The smallest singular value of inhomogeneous random matrices and efficient net estimates

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Point Distribution Webinar

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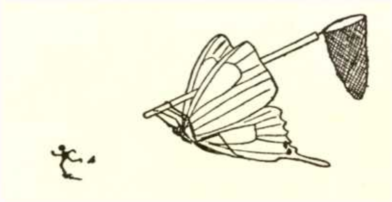


a) Pick a large integer, say, $n = 1000000$. Flip a fair die n^2 times. 
Fill an $n \times n$ matrix with the outcomes. How likely is this matrix to be invertible?

$$\begin{bmatrix} 1 & 2 & 6 & 4 & 2 & 5 \\ 3 & 1 & 5 & 3 & 3 & 6 \\ 2 & 3 & 6 & 5 & 1 & 1 \\ 1 & 3 & 2 & 6 & 2 & 5 \\ 4 & 3 & 6 & 1 & 4 & 2 \\ 2 & 3 & 3 & 6 & 4 & 5 \end{bmatrix}$$

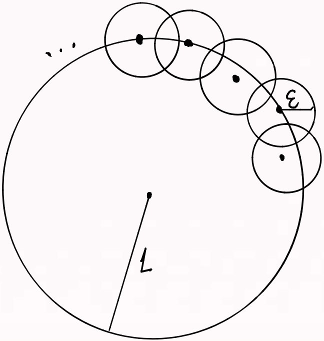
b) And what if now we do not roll the same die every time, but rather use *different* dice to determine different entries?

Notation

- $\mathbb{R}^n$ – euclidean n-dimensional space with standard basis $e_1, ..., e_n$;
- B_2^n – euclidean unit ball in $\mathbb{R}^n$;
- $\mathbb{S}^{n-1}$ – unit sphere in $\mathbb{R}^n$;
- $|x| = \sqrt{x_1^2 + ... + x_n^2}$;
- The Hilbert-Schmidt norm of a matrix A is $||A||_{HS} = \sqrt{\sum_{i,j} a_{ij}^2}$;
- Singular values of A are the axi of the ellipsoid AB_2^n, denoted $\sigma_1(A) \geq ... \geq \sigma_n(A)$;
- The operator norm $||A|| = \sup_{x \in \mathbb{S}^{n-1}} |Ax| = \sigma_1(A)$;
- The smallest singular value $\sigma_n(A) = \inf_{x \in \mathbb{S}^{n-1}} |Ax|$;
- A random variable ξ is anti-concentrated if

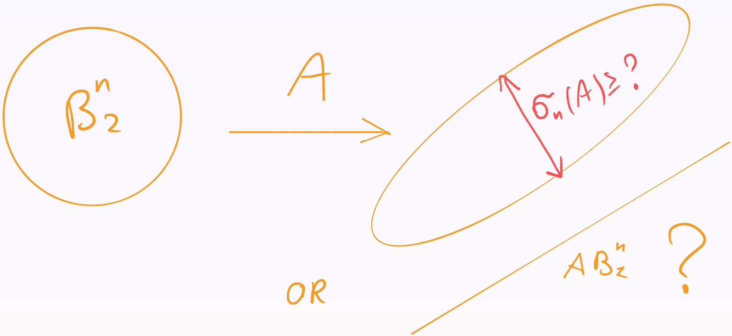
$\sup_{z \in \mathbb{R}} P(|\xi - z| < 1) < b \in [0, 1)$.

Recall: there exists a Euclidean epsilon-net N on the unit sphere of cardinality $< (3/\varepsilon)^n$.



Main question

Question: how likely is a random $n \times n$ matrix A to be invertible?



A harder question: how likely is the smallest singular value $\sigma_n(A) = \inf_{x \in \mathbb{S}^{n-1}} |Ax|$ to be bigger than  ?

History

A is an $n \times n$ Gaussian, with i.i.d. entries $a_{ij} \sim N(0,1)$

$$\sigma_n(A) \approx \frac{1}{\sqrt{n}}.$$

Furthermore, for every $\epsilon \in (0,1)$,

$$P\left(\sigma_n(A) \leq \frac{\epsilon}{\sqrt{n}}\right) \leq C\epsilon.$$

(Edelman, Szarek independently in 1990s)

History

A is $n \times n$ matrix with i.i.d. Bernoulli ± 1 entries

Conjecture (Erdos) 1950s: $P(\sigma_n(A) = 0) = Cn^2 \cdot 2^{-n}$
(when a pair of columns or rows coincide, and rarely elsewhere)

- Kolmos 60s: $P(\sigma_n(A) = 0) = o(1)$;
- Khan, Kolmos, Szemeredi 1995: $P(\sigma_n(A) = 0) \leq 0.99^n$;
- Tao, Vu 2006, 2007: $P(\sigma_n(A) = 0) \leq 0.75^n$;
- Bourgain, Vu, Wood, 2010: $P(\sigma_n(A) = 0) \leq \sqrt{2}^{-n}$;
- Tikhomirov, 2019: $P(\sigma_n(A) = 0) \leq (0.5 + o(1))^n$!

History

A random variable ξ is *sub-Gaussian* if for all $t > 0$,

$$P(|\xi| \geq t) \leq e^{-Kt^2}.$$

A is $n \times n$, has entries a_{ij} i.i.d. sub-Gaussian, $\mathbb{E}a_{ij} = 0$, $\mathbb{E}a_{ij}^2 = 1$

Rudelson, Vershynin 2008:

$$P\left(\sigma_n(A) \leq \frac{\epsilon}{\sqrt{n}}\right) \leq C\epsilon + e^{-cn}.$$

Note: this combines the behavior of Gaussian matrices and the Bernoulli ± 1 matrices.

A is $n \times n$, has entries a_{ij} uniformly anti-concentrated, i.i.d., $\mathbb{E}a_{ij} = 0$, $\mathbb{E}a_{ij}^2 = 1$

Rebrova, Tikhomirov 2016:

~~Sub-Gaussian~~

$$P\left(\sigma_n(A) \leq \frac{\epsilon}{\sqrt{n}}\right) \leq C\epsilon + e^{-cn}.$$

History

A is $n \times n$, has independent UAC entries, $\mathbb{E}||A||_{HS}^2 \le Kn^2$, i.i.d. rows

L, 2018+

$$P\left(\sigma_n(A) \le \frac{\epsilon}{\sqrt{n}}\right) \le C\epsilon + e^{-cn}.$$

~~i.i.d columns~~
~~mean zero~~
~~$\mathbb{E}a_{ij}^2 = 1$~~

Remark

In fact, it is enough to assume for any $p > 0$,

$$\sum_{i=1}^{n}\left(\mathbb{E}|Ae_i|^{2p}\right)^{\frac{1}{p}} \le Kn^2; \quad \sum_{i=1}^{n}\left(\mathbb{E}|A^T e_i|^{2p}\right)^{\frac{1}{p}} \le Kn^2.$$

Note: in principle, all entries may have infinite second moment, but distribution has to depend on n.

Bai, Cook, Edelman, Gordon, Guedon, Huang, Koltchinckii, Latala, Litvak, Lytova, Meckes, Meckes, Mendelson, Pajor, Paouris, Rebrova, Rudelson, O'Rourke, Szarek, Tao, Tatarko, Tomczak-Jaegermann, Tikhomirov, Van Handel, Vershynin, Vu, Yaskov, Yin, Youssef,...

Theorem (L, Tikhomirov, Vershynin 2019+)

Let A be an $n \times n$ random matrix with

- independent entries a_{ij}
- $\mathbb{E}||A||_{HS}^2 \leq Kn^2$
- a_{ij} are UAC, that is $\sup_{z\in\mathbb{R}} P(|a_{ij} - z| < 1) < b \in (0,1)$

Then for every $\epsilon \in (0,1)$,

$$P\left(\sigma_n(A) < \frac{\epsilon}{\sqrt{n}}\right) \leq C\epsilon + e^{-cn},$$

where C and c are absolute constants which depend (polynomially) only on K and b.

Arbitrary aspect ratio: history

Question: what if A is an $N \times n$ random matrix with $N \geq n$?

Litvak, Pajor, Rudelson, Tomczak-Jaegermann, 2005

$N \geq n + \frac{n}{C \log n}$, strong assumptions: $P(\sigma_n(A) \leq C_1\sqrt{N}) \leq e^{-C_2 N}$.

Rudelson, Vershynin, 2009

$N \geq n$, a_{ij} i.i.d. sub-Gaussian, $\mathbb{E}a_{ij} = 0$, $\mathbb{E}a_{ij}^2 = 1$. Then for any $\epsilon \in (0,1)$,

$$P\left(\sigma_n(A) \leq \epsilon(\sqrt{N+1} - \sqrt{n})\right) \leq C_1 \epsilon^{N-n+1} + e^{-C_2 N};$$

Tao, Vu, 2010

Replaced sub-Gaussian with $\mathbb{E}a_{ij}^{C_1} \leq 1$, but $N \in [n, n + C_2]$

Vershynin, 2011

Replaced sub-Gaussian with $\mathbb{E}a_{ij}^4 < \infty$ but

$$P\left(\sigma_n(A) \leq \epsilon(\sqrt{N+1} - \sqrt{n})\right) \leq \delta(\epsilon) \to_{\epsilon \to 0} 0.$$

Theorem (L. 2018+)

Let $N \geq n \geq 1$ be integers. Let A be an $N \times n$ random matrix with

- independent UAC entries a_{ij}
- i.i.d. rows
- $\mathbb{E}a_{ij} = 0$
- $\mathbb{E}a_{ij}^2 = 1$.

Then for every $\epsilon > 0$,

$$P\left(\sigma_n(A) < \epsilon(\sqrt{N+1} - \sqrt{n})\right) \leq (C\epsilon \log 1/\epsilon)^{N-n+1} + e^{-cN},$$

where C and c are absolute constants which depend (polynomially) only on the concentration function bounds.

Remark: a more general result in fact follows...

Proposition 1 (L. 2018+) tall case with dependent columns

Suppose A is an $N \times n$ random matrix with independent rows, $\mathbb{E}||A||_{HS}^2 \leq KNn$, $N \geq C_0 n$, and assume for every $x \in \mathbb{S}^{n-1}$,

$$\sup_{y \in \mathbb{R}} P(|\langle A^T e_i, x\rangle - y| \leq 1) \leq b \in (0,1).$$

Then

$$\mathbb{E}\sigma_n(A) \geq c\sqrt{N}.$$

Proposition 2 (L. 2018+) tall case with low moments

Fix $p > 0$. Suppose $N \geq C_0' n$, A is an $N \times n$ random matrix with independent UAC entries. Suppose

$$\sum_{i=1}^{n} \left(\mathbb{E}|Ae_i|^{2p}\right)^{\frac{1}{p}} \leq KnNe^{\frac{c_0 N}{n}}.$$

Then

$$P(\sigma_n \leq C_1\sqrt{N}) \leq e^{-C_2 \min(p,1) N}.$$

A naive attempt

Goal: $P(\sigma_n(A) \leq 2\heartsuit) \leq \diamondsuit$.

Discretize $\mathbb{S}^{n-1}$:

Suppose we find a small finite set $\mathcal{N} \subset \mathbb{R}^n$ with

- $\#\mathcal{N} \leq \spadesuit$;
- $\forall x \in \mathbb{S}^{n-1}\, \exists y \in \mathcal{N} : |A(x-y)| \leq \heartsuit$ with probability $\geq 1 - \clubsuit$.

Then we write:

$$P(\sigma_n(A) \leq \heartsuit) = P\left(\inf_{x \in \mathbb{S}^{n-1}} |Ax| \leq \heartsuit\right) \leq$$

$$P\left(\inf_{y \in \mathcal{N}} |Ay| \leq 2\heartsuit\right) + \clubsuit = P\left(\exists y \in \mathcal{N} : \ |Ax| \leq 2\heartsuit\right) + \clubsuit \leq$$

$$\spadesuit \cdot \sup_{y \in \mathcal{N}} P(|Ay| \leq 2\heartsuit) + \clubsuit.$$

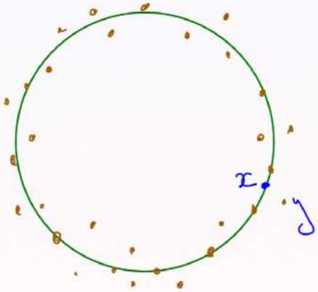
So if we know that for each y, $P(|Ay| \leq 2\heartsuit) \leq \frac{\diamondsuit - \clubsuit}{\spadesuit}$, we are done!

The net result

Theorem (L. 2018+) – Lite version

There exists a deterministic net $\mathcal{N} \subset \frac{3}{2}B_2^n \setminus \frac{1}{2}B_2^n$ of cardinality 1000^n such that for any integer N and any $N \times n$ random matrix A with independent columns, with probability at least $1 - e^{-5n}$, for every $x \in \mathbb{S}^{n-1}$ there exists $y \in \mathcal{N}$ such that

$$|A(x-y)| \leq \frac{100}{\sqrt{n}}\sqrt{\mathbb{E}||A||_{HS}^2}.$$



- 1000^n points
- whp,

$|A(x-y)| \leq$ small

Previously known cases

Folklore: A has sub-gaussian independent entries a_{ij}, $\mathbb{E}a_{ij}=0$, $\mathbb{E}a_{ij}^2=const$.

- Let $\mathcal{N}$ be the standard ε-net, i.e. such that

$$\mathbb{S}^{n-1} \subset \cup_{x\in\mathcal{N}} \left(x+\varepsilon B_2^n\right),$$

and $\#\mathcal{N} \leq \left(\frac{3}{\varepsilon}\right)^n$.

- Then we can estimate $|A(x-y)| \leq ||A||\varepsilon \leq C\frac{||A||_{HS}\varepsilon}{\sqrt{n}}$?
- Recall, for any matrix A: $\frac{1}{\sqrt{n}}||A||_{HS} \leq ||A|| \leq ||A||_{HS}$.
- But specifically for sub-gaussian mean zero variance 1 case,

$$P\left(||A|| \geq \frac{100}{\sqrt{n}}\sqrt{\mathbb{E}||A||_{HS}^2}\right) \leq e^{-5n}. \quad (1)$$

- Without strong assumptions, (1) is not true.

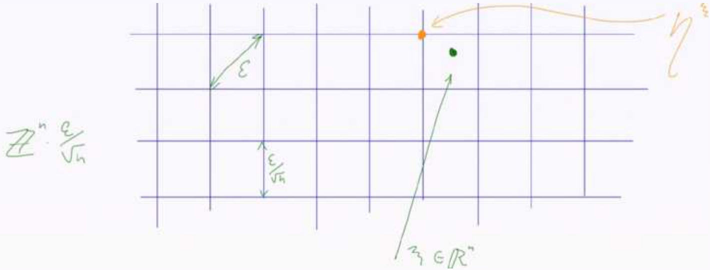
Previously known cases

Rebrova, Tikhomirov (2016) proved this Theorem assuming i.i.d. entries a_{ij}, with $\mathbb{E}a_{ij} = 0$, $\mathbb{E}a_{ij}^2 = const$, and $N = n$.

Guedon, Litvak, Tatarko (2018) extended the result of Rebrova and Tikhomirov in the case of arbitrary n, N, and replaced i.i.d. entries with i.i.d. columns.

- Advantage: the Theorem only assumes independence of columns, and no other structural assumptions!
- In particular, allowing dependent columns is crucial for the proof of the arbitrary aspect ratio result.

Randomized rounding (Raghavan-Tompson 1987, Beck 1987, Kannan-Vempala 1997, Srinivasan 1999, Alon-Klartag 2017, Klartag-L 2018+, L 2018+, Tikhomirov 2019+,...)



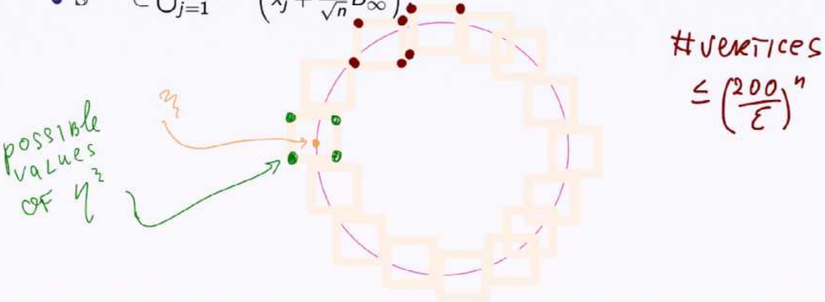
Definition

For $\xi \in \mathbb{S}^{n-1}$, write each $\xi_i = \frac{\epsilon}{\sqrt{n}}(k_i + p_i)$ for $k_i \in \mathbb{Z}$ and $p_i \in [0,1)$. Consider a random vector $\eta^\xi \in (\epsilon/\sqrt{n})\mathbb{Z}^n$:

$$\eta_i^\xi = \begin{cases} \frac{\epsilon}{\sqrt{n}} k_i, & \text{with probability } 1 - p_i \\ \frac{\epsilon}{\sqrt{n}}(k_i + 1), & \text{with probability } p_i. \end{cases}$$

Proof – step 1: comparison via Hilbert-Schmidt

- $\mathbb{S}^{n-1} \subset \bigcup_{j=1}^{(100/\epsilon)^n} \left(x_j + \frac{\epsilon}{\sqrt{n}} B_\infty^n \right)$



- Therefore, there is a set $\mathcal{N}$ such that for all $\xi \in \mathbb{S}^{n-1}$, we have $\eta^\xi \in \mathcal{N}$, and $\#\mathcal{N} \leq \left(\frac{100}{\epsilon}\right)^n$;
- We have $\|\xi - \eta^\xi\|_\infty \leq \frac{\epsilon}{\sqrt{n}}$ and $\mathbb{E}\eta^\xi = \xi$;
- Hence, using the fact that $\mathbb{E}(\eta^\xi - \xi) = 0$, we get:

$$\mathbb{E}|\langle \eta^\xi - \xi, \theta \rangle|^2 \leq \frac{\epsilon^2 |\theta|^2}{n} \quad (\heartsuit)$$

Proof – step 1: comparison via Hilbert-Schmidt

Lemma 1 (comparison via Hilbert-Schmidt)

There exists a collection of points $\mathcal{F}$ with $\#\mathcal{F} \leq (\frac{C}{\epsilon})^{n-1}$ such that for any (deterministic) matrix $A : \mathbb{R}^n \to \mathbb{R}^N$, for every $\xi \in \mathbb{S}^{n-1}$ there exists an $\eta \in \mathcal{F}$ satisfying

$$|A(\eta - \xi)| \leq \frac{\epsilon}{\sqrt{n}} ||A||_{HS}.$$

Proof.

- Recall: $|Ax|^2 = \sum_{i=1}^N \langle A^T e_i, x\rangle^2$, where $A^T e_i$ are the rows of A;
- By ($\heartsuit$), $\mathbb{E}_\eta |\langle \eta^\xi - \xi, A^T e_i\rangle|^2 \leq C \frac{\epsilon^2 |A^T e_i|^2}{n}$;
- Summing up, we get

$$\mathbb{E}_\eta |A(\eta^\xi - \xi)|^2 = \mathbb{E}_\eta \sum_{i=1}^N \langle A^T e_i, \eta^\xi - \xi\rangle^2 \leq \left(C' \frac{\epsilon}{\sqrt{n}} ||A||_{HS} \right)^2;$$

- If $P(\textit{find a red ball in a box}) \geq 0.1$ then there exists a red ball in a box.

□

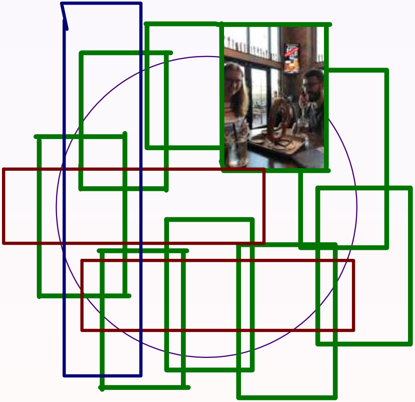
Proof – step 2: parallelepipeds

Remark

$$P(||A||_{HS}^2 \geq 10\mathbb{E}||A||_{HS}^2) \leq 0.1.$$

Thus Lemma 1 implies the Theorem with probability 0.9 rather than $1 - e^{-5n}$. Not good:(

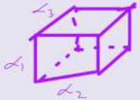
Idea of Rebrova and Tikhomirov, 2016: cover with parallelepipeds and not just cubes!



Proof – step 2: parallelepipeds

Admissible set of parallelepipeds

- For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_i > 0$, we fix the parallelepiped



$$P_\alpha = \{x \in \mathbb{R}^n : |x_i| \le \alpha_i\}.$$

- For $\kappa > 1$, denote $\Omega_\kappa = \left\{\alpha \in \mathbb{R}^n : \alpha_i \in [0,1], \prod_{i=1}^n \alpha_i > \kappa^{-n}\right\}$.
- Note: if $\alpha \in \Omega_\kappa$ then $P_\alpha \ge (0.5\kappa)^{-n}$ – hence the covering is not too big.

Lemma 2 (comparison via parallelepipeds)

Pick any $\alpha \in \Omega_\kappa$. Let A be any $N \times n$ matrix. There exists a net $\mathcal{F}_\alpha$ with $\#\mathcal{F}_\alpha \le \left(\frac{100\kappa}{\epsilon}\right)^n$ such that for every $\xi \in \mathbb{S}^{n-1}$ there exists an $\eta \in \mathcal{F}_\alpha$ satisfying

$$|A(\eta - \xi)| \le \frac{\epsilon}{\sqrt{n}} \sqrt{\sum_{i=1}^n \alpha_i^2 |Ae_i|^2}.$$

Proof – step 3: $\mathcal{B}_\kappa$ and nets on nets

Key definition: for any matrix A

$$\mathcal{B}_\kappa(A) := \min_{\alpha_i \in [0,1],\ \prod_{i=1}^n \alpha_i \geq \kappa^{-n}} \sum_{i=1}^n \alpha_i^2 |Ae_i|^2.$$

minimum cuts off heavy tails!

$\|A\|_{HS}^2$

Corollary of Lemma 2

Let A be any $N \times n$ matrix. There exists a small enough net $\mathcal{F}$ such that for every $\xi \in \mathbb{S}^{n-1}$ there exists an $\eta \in \mathcal{F}$ satisfying

$$|A(\eta - \xi)| \leq \frac{\epsilon}{\sqrt{n}} \sqrt{\mathcal{B}_\kappa(A)}.$$

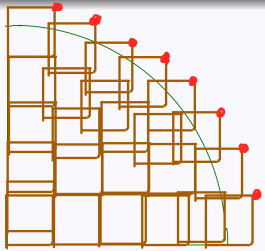
But the net depends on the matrix! Not good:(

Proof – step 3: $\mathcal{B}_\kappa$ and nets on nets

Way out: discretize the admissible set Ω_κ.

$$T\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} = \begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix}$$

$$a_j = \sqrt{\frac{\log 1/\alpha_j}{n\log\kappa}}$$



$$\Omega_\kappa = \left\{\alpha : \alpha_i \in [0,1],\ \prod_{i=1}^{n}\alpha_i \ge \kappa^{-n}\right\}$$

$$T\Omega_\kappa = \left\{a : a_i \ge 0,\ \sum_{i=1}^{n} a_i^2 \le 1\right\}$$

(only need a few cubes to cover the ball!)

The "nets on nets" Lemma

There exists a collection $\mathcal{F} \subset \Omega_{\kappa^2}$ of cardinality 30^n such that for any $\alpha \in \Omega_\kappa$ there exists a $\beta \in \mathcal{F}$ so that for all $i = 1, \ldots, n$ we have $\alpha_i^2 \geq \beta_i^2$.
In particular, for any $N \times n$ matrix A, we have

$$\mathcal{B}_\kappa(A) \geq \min_{\beta\in\mathcal{F}} \sum_{i=1}^{n} \beta_i^2 |Ae_i|^2.$$

A separate discussion: on the minimal dispersion

Minimal dispersion: definition.

The set of all axis-parallel boxes contained in the unit cube is denoted by R_d, that is

$$R_d := \left\{ \prod_{i=1}^{d} I_i \mid I_i = [a_i, b_i) \subset [0,1] \right\}.$$

Given a finite set $P \subset [0,1]^d$, its dispersion is defined as

$$d(P) = \sup\{|B| \mid B \in R_d,\, B \cap P = \emptyset\}.$$

The minimal dispersion is defined as the function of two variables n and d as

$$d^*(n,d) = \inf_{\substack{P \subset [0,1]^d \\ |P|=n}} (P).$$

Its inverse function is

$$N(\epsilon, d) = \min\{n \in \mathbb{N} \mid d^*(n,d) \leq \epsilon\}.$$

In other words, $N(\epsilon, d)$ is the smallest number of points inside the unit cube such that each axis-parallel box of volume ϵ contains at least one point of this collection.

Minimal dispersion: definition.

Theorem (Litvak, Livshyts)

Let $d \geq 2$ and $\epsilon \in (0, 1/2]$. Then

$$N(\epsilon, d) \leq 12e \, \frac{4d \log\log(8/\epsilon) + \log(1/\epsilon)}{\epsilon}.$$

Moreover, the random choice of points with respect to the uniform distribution on the cube gives the result with high probability.

Rote-Tichy'96; Larcher'17; Bukh-Chao'21; Dumitrescu-Jiang'13; Aistleitner-Hinrichs-Rudolf'17; Blumer-Ehrenfeucht-Hausser-Warmuth'89; Rudolf'18; Sosnovec'18; Ulrich-Vybiral'18; MacKay'21; Litvak'20; Hinrichs-Krieg-Kunsch-Rudolf'20

Methods

δ-approximation for $\mathcal{B}_d(\varepsilon)$ – definition

Given $0 < \delta \leq \varepsilon \leq 1$ we say that $\mathcal{N} \subset \mathcal{R}_d$ is a δ-approximation for $\mathcal{B}_d(\varepsilon)$ if for every $B \in \mathcal{B}_d(\varepsilon)$ there exists $B_0 \in \mathcal{N}$ such that $B_0 \subset B$ and

$$|B_0| \geq \delta.$$

We define a δ-approximation for $\mathcal{B}_d^0(\varepsilon)$ and $\widetilde{\mathcal{B}}_d(\varepsilon)$ in a similar way.

Lemma (Rudolf/Litvak)

Let $d \geq 1$ and $\varepsilon, \delta \in (0,1)$. Let $\mathcal{N}$ be a δ-approximation for $\mathcal{B}_d(\varepsilon)$ and let $\widetilde{\mathcal{N}}$ be a δ-approximation for $\widetilde{\mathcal{B}}_d(\varepsilon)$. Assume both $|\mathcal{N}| \geq 3$ and $|\widetilde{\mathcal{N}}| \geq 3$. Then

$$N(\varepsilon, d) \leq \frac{3 \ln |\mathcal{N}|}{\delta} \qquad \text{and} \qquad \widetilde{N}(\varepsilon, d) \leq \frac{3 \ln |\widetilde{\mathcal{N}}|}{\delta}.$$

Moreover, the random choice of independent points (with respect to the uniform distribution on Q_d) gives the result with probability at least $1 - 1/|\mathcal{N}|$.

Methods

Lemma (Livshyts, Litvak) – upgrade on Step 3

Let $d \geq 2$ be an integer, $\epsilon \in (0,1)$, and $\gamma > 0$. Let $\delta = \epsilon^{1+\gamma}$. Then the size of an optimal $(\epsilon^{1+\gamma})$-approximation of $\mathcal{B}_d^0(\varepsilon)$ equals to

$$N(S_{d-1}, -\gamma S_{d-1}) \leq 7d \ln d \left(\frac{1+\gamma}{\gamma}\right)^{d-1},$$

where S_{d-1} is a regular $(d-1)$-dimensional simplex.

Corollary (via shifts)

Let $d \geq 2$ be an integer, $\varepsilon \in (0,1)$, and $\gamma > 0$. Let $\delta = \varepsilon^{1+\gamma}/4$. There exists a δ-approximation $\mathcal{N}$ for $\mathcal{B}_d(\varepsilon)$ of cardinality at most

$$7d \ln d \, \frac{(1+1/\gamma)^d (\ln(e/\varepsilon^{1+\gamma}))^d}{\varepsilon^{1+\gamma}}.$$

Back to random matrices...

A net for deterministic matrices: combining steps 1-3.

Theorem about deterministic matrices

There exists a deterministic net $\mathcal{N}$ of cardinality 1000^n such that for any integer N and any $N \times n$ deterministic matrix A, for every $x \in \mathbb{S}^{n-1}$ there exists $y \in \mathcal{N}$ such that

$$|A(x-y)| \leq \frac{100}{\sqrt{n}} \sqrt{\mathcal{B}_{10}(A)}.$$

This reduces the proof of the Theorem to estimating the large deviation of $\mathcal{B}_\kappa(A)$ when A is a random matrix coming from an appropriate model.

Step 4: Large deviation of $\mathcal{B}_\kappa$.

Lemma

Let A be a random matrix with independent columns. Pick any $\kappa > 1$. Then

$$P\left(B_\kappa(A) \geq 10\mathbb{E}||A||_{HS}^2\right) \leq (C\kappa)^{-2n}.$$

Proof.

- Denote $Y_i = |Ae_i|$. If $B_\kappa(A) \geq 10\sum_{i=1}^n \mathbb{E}Y_i^2$, then for any collection $\alpha_1, ..., \alpha_n \in [0,1]$, either

$$\sum_{i=1}^n \alpha_i^2 Y_i^2 \geq 10 \sum_{i=1}^n \mathbb{E}Y_i^2,$$

or

$$\prod_{i=1}^n \alpha_i < \kappa^{-n}.$$

- Consider a collection of random variables $\alpha_i^2 = \min\left(1, \frac{\mathbb{E}Y_i^2}{Y_i^2}\right)$.

Step 4: Large deviation of $\mathcal{B}_\kappa$.

Proof.

- We estimate

$$P\left(B_\kappa(A) \geq 10\mathbb{E}||A||_{HS}^2\right) \leq$$

$$P\left(\sum_{i=1}^{n} \min\left(1, \frac{\mathbb{E}Y_i^2}{Y_i^2}\right) Y_i^2 \geq 10\mathbb{E}||A||_{HS}^2\right) +$$

$$P\left(\prod_{i=1}^{n} \min\left(1, \frac{\mathbb{E}Y_i^2}{Y_i^2}\right) < \kappa^{-2n}\right) =: P_1 + P_2.$$

- $P_1 = 0$.
- By Markov's inequality, $P_2 \leq (C\kappa)^{-2n}$.

$\square$

Summary: the non-lite version

Theorem (NON-lite)

Fix $n \in \mathbb{N}$. Consider any $S \subset \mathbb{R}^n$. Pick any $\gamma \in (1, \sqrt{n})$, $\epsilon \in (0, \frac{1}{20\gamma})$, $\kappa > 1$, $p > 0$ and $s > 0$.
There exists a (deterministic) net $\mathcal{N} \subset S + 4\epsilon\gamma B_2^n$, with

$$\#\mathcal{N} \leq \begin{cases} N(S, \epsilon B_2^n) \cdot (C_1\gamma)^{\frac{C_2 n}{\gamma^{0.08}}}, & \text{if } \log\kappa \leq \frac{\log 2}{\gamma^{0.09}}, \\ N(S, \epsilon B_2^n) \cdot (C\kappa \log\kappa)^n, & \text{if } \log\kappa \geq \frac{\log 2}{\gamma^{0.09}}, \end{cases}$$

such that for every $N \in \mathbb{N}$ and every random $N \times n$ matrix A with independent columns, with probability at least

$$1 - \kappa^{-2pn} \left(1 + \frac{1}{s^p}\right)^n,$$

for every $x \in S$ there exists $y \in \mathcal{N}$ such that

$$|A(x-y)| \leq C_3 \frac{\epsilon\gamma\sqrt{s}}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} (\mathbb{E}|Ae_i|^{2p})^{\frac{1}{p}}}.$$

Here C, C_1, C_2, C_3 are absolute constants. γ is the "sparsity" parameter

Thanks for your attention!