Diameter bounded equal measure partitions of Ahlfors regular metric measure spaces

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For presentation at
Point Distributions Webinar
Florida State University
Joint work with Giacomo Gigante, University of Bergamo.

9 December 2020

Acknowledgements

Thanks to the Università degli studi di Bergamo for hosting me for a one month visit in 2015, funded by the project “Progetto ITALY®- Azione 3: Grants for Visiting Professor and Scholar”.

Part of the work was conducted while I was attending the Program on “Minimal Energy Point Sets, Lattices, and Designs” at the Erwin Schrödinger International Institute for Mathematical Physics, 2014, as a Visiting Fellow of the Australian National University.

The work was completed while I was sessional staff at the University of Newcastle, Australia, in 2015.

Pozible supporters 2014

$32+ SvA, AD, CF, OF, KM, JP, AHR, RR, ES, MT.

$50+ Yvonne Barrett, Angela M. Fearon, Sally Greenaway, Dennis Pritchard, Susan Shaw, Bronny Wright.

$64+ Naomi Cole.

$100+ Russell family, Jonno Zilber.

$128+ Jennifer Lanspeary, Vikram.

...and others who did not want acknowledgement.

Main result

Theorem 1

Let (X, ρ, μ) be a connected Ahlfors regular metric measure space of dimension d and finite measure.

Then there exist positive constants c_3 and c_4 such that for every sufficiently large N, there is a partition of X into N regions of measure $\mu(X)/N$, each contained in a ball of radius $c_3 N^{-1/d}$ and containing a ball of radius $c_4 N^{-1/d}$.

Outline of talk

- Applications to numerical integration
- Precedents: Equal area partitions of the unit sphere
- Dyadic cubes on Ahlfors regular spaces
- Construction of the partition

Error bounds for positive weight quadrature rules

Let X be a smooth compact d-dimensional Riemannian manifold without boundary. For any $1 \leqslant p \leqslant +\infty$, $\alpha > d/p$ and $\kappa \geqslant 1/2$, for large enough N there exists a quadrature rule with points $\{z_j\}_{j=1}^N$ and positive weights $\{\omega_j\}_{j=1}^N$ such that

$$\left| \sum_{j=1}^{N} \omega_j f(z_j) - \int_X f(x) d\mu(x) \right| \leqslant c N^{-\alpha/d} \|f\|_{W^{\alpha,p}}$$

for all functions $f \in W^{\alpha,p}$, the Sobolev class of functions f with $(I + \Delta)^{\alpha/2} f \in L^p(X)$.

(Brandolini et al. 2013; Brandolini et al. 2014)

Equal area partition of the unit sphere

Stolarsky (1973) asserted the existence for any natural number N of a partition of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ into N regions of equal volume and diameter bounded as order $\mathrm{O}(N^{-1/d})$.

Feige and Schechtman (2002) gave a construction using a directed tree of Voronoi cells that can be modified to satisfy Stolarsky's assertion (L 2007).

This construction could be further modified to yield an equal measure partition of a compact connected Riemannian manifold (L 2014).

(Stolarsky 1973; Feige and Schechtman 2002; L 2007; L 2014)

The unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$

Definition 2

The unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is

$$\mathbb{S}^d := \left\{ x \in \mathbb{R}^{d+1} \;\middle|\; \sum_{k=1}^{d+1} x_k^2 = 1 \right\}.$$

Equal-area partitions of $\mathbb{S}^d$

Definition 3

An *equal area partition* of $\mathbb{S}^d$ is a nonempty finite set $\mathcal{P}$ of Lebesgue measurable subsets of $\mathbb{S}^d$, such that

$$\bigcup_{R \in \mathcal{P}} R = \mathbb{S}^d,$$

and for each $R \in \mathcal{P}$,

$$\sigma(R) = \frac{\sigma(\mathbb{S}^d)}{|\mathcal{P}|},$$

where σ is the Lebesgue area measure on $\mathbb{S}^d$.

Diameter bounded sets of partitions

Definition 4

The *diameter* of a region $\boldsymbol{R} \subset \mathbb{R}^{d+1}$ is defined by

$$\operatorname{diam} \boldsymbol{R} := \sup\{\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}\},$$

where $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y})$ is the $\mathbb{R}^{d+1}$ Euclidean distance $\|\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}\|$.

Definition 5

A set Ξ of partitions of $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is *diameter-bounded* with *diameter bound* $\boldsymbol{K} \in \mathbb{R}_+$
if for all $\mathcal{P} \in \Xi$, for each $\boldsymbol{R} \in \mathcal{P}$,

$$\operatorname{diam} \boldsymbol{R} \leqslant \boldsymbol{K} \, |\mathcal{P}|^{-1/d}.$$

Partitions of $\mathbb{S}^2$, $\mathbb{S}^3$ and $\mathbb{S}^d$

Alexander (1972) uses the existence of a diameter-bounded set of equal-area partitions of $\mathbb{S}^2$ to analyze the maximum sum of distances between points, and suggests a construction for this set. Alexander (1972) suggests a construction different from Zhou (1995).

Zhou (1995) gives a construction for $\mathbb{S}^2$. Saff (2003) and by Sloan (2003) modify this to give a construction for $\mathbb{S}^3$, which generalizes to the EQ construction for $\mathbb{S}^d$ (L 2007).

(Alexander 1972; Zhou 1995; Saff 2003; Sloan 2003; L 2007)

Stolarsky's assertion on $\mathbb{S}^d$

Stolarsky (1973) asserts the existence of a diameter-bounded set of equal-area partitions of $\mathbb{S}^d$ for all d, but offers no construction or existence proof.

Beck and Chen (1987) quotes Stolarsky. Bourgain and Lindenstrauss (1988) quotes Beck and Chen.

Wagner (1993) implies the existence of an EQ-like construction for $\mathbb{S}^d$. Bourgain and Lindenstrauss (1993) gives a partial construction.

Feige and Schechtman (2002) gives a construction which when modified (L 2007) proves Stolarsky's assertion.

(Stolarsky 1973; Beck and Chen 1987; Bourgain and Lindenstrauss 1988, 1993; Wagner 1993; Feige and Schechtman 2002; L 2007)

Spherical caps

The *spherical cap* $\boldsymbol{S}(\boldsymbol{p}, \theta) \in \mathbb{S}^{\boldsymbol{d}}$ is

$$\boldsymbol{S}(\boldsymbol{p}, \theta) := \left\{ \boldsymbol{q} \in \mathbb{S}^{\boldsymbol{d}} \mid \underline{\boldsymbol{p}} \cdot \underline{\boldsymbol{q}} \geqslant \cos(\theta) \right\}.$$

For $\boldsymbol{d} > \mathbf{1}$, the area of a spherical cap of spherical radius θ is

$$\mu(\theta) := \boldsymbol{\sigma}\big(\boldsymbol{S}(\boldsymbol{p}, \theta)\big) = \omega \int_{\mathbf{0}}^{\theta} (\sin \xi)^{\boldsymbol{d}-\mathbf{1}} \boldsymbol{d}\xi,$$

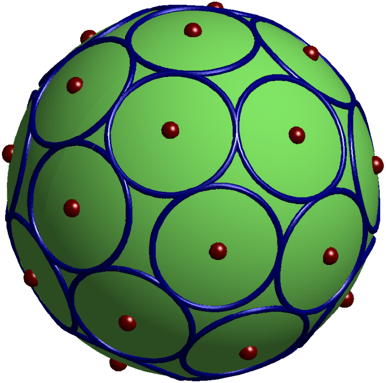
where $\omega = \boldsymbol{\sigma}(\mathbb{S}^{\boldsymbol{d}-\mathbf{1}})$.

Outline of the modified Feige-Schechtman algorithm

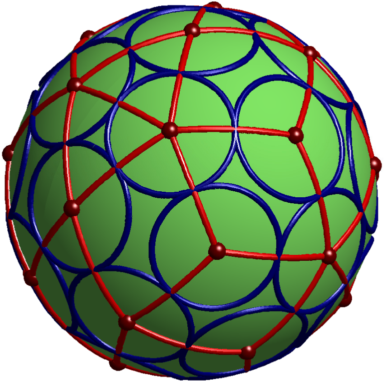
1. Find spherical radius θ_c of caps with $\mu(\theta) = \sigma(\mathbb{S}^d)/N$
2. Create an optimal packing of caps of spherical radius θ_c
3. Create a graph of kissing caps
4. Create a directed tree from graph
5. Create a Voronoi tessellation
6. Move area from V-cells towards the root of the tree
7. Split adjusted cells

(Feige and Schechtman 2002; L 2007)

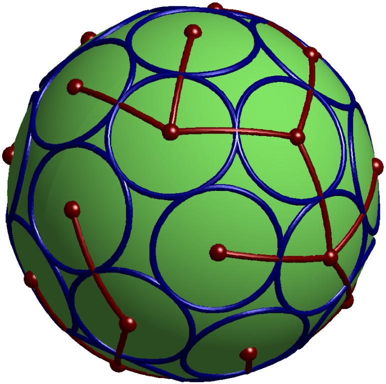
2. Create optimal packing of caps



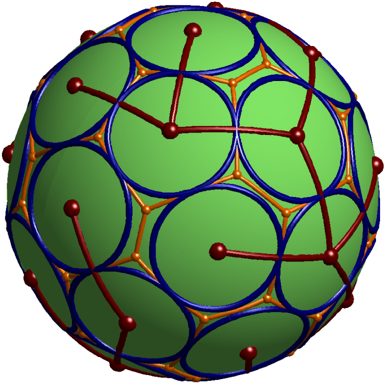
3. Create graph of kissing caps



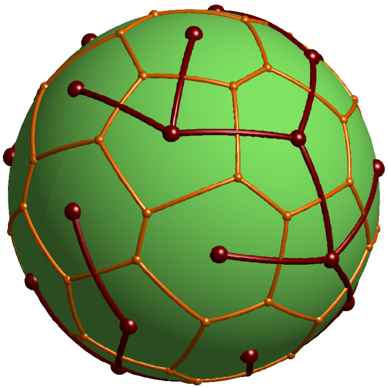
4. Create directed tree from graph



5. Create Voronoi tessellation



6. Move area from V-cells towards root



Outline of proof the F-S bound

- Packing radius is $\theta_c = \mathrm{O}(N^{-1/d})$.
- V-cells are in caps of spherical radius $2\theta_c$.
- Each V-cell has area larger than target area.
- Area is moved from V-cells of kissing packing caps.
- Adjusted cells are in caps of spherical radius $4\theta_c$.
- So Euclidean diameter is bounded above by

$$8\theta_c = \mathrm{O}(N^{-1/d}).$$

(Feige and Schechtman 2002; L 2007)

Ahlfors regular space

Definition 6

An Ahlfors regular metric measure space of dimension $d > 0$ is a complete metric space X with a Borel measure μ with positive constants c_1 and c_2 such that all open metric balls $B(x, r)$ with $x \in X$, $0 < r \leqslant \operatorname{diam}(X)$ satisfy the bounds

$$c_1 r^d \leqslant \mu(B(x, r)) \leqslant c_2 r^d.$$

(David and Semmes 1997; Gromov 2007)

Dyadic cubes

Definition 7

A collection of open subsets of X, $\{Q_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$ is a family of dyadic cubes of X if there exist three constants $\delta \in (0,1)$ and $0 < a_0 \leqslant a_1$, with the following properties:

$$\mu\Big(X \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k\Big) = 0 \quad \text{for all } k. \tag{1}$$

$$Q_\alpha^k \cap Q_\beta^k = \emptyset \quad \text{for each } k \text{ and } \alpha \neq \beta. \tag{2}$$

$$\text{If } \ell > k \text{ then either } Q_\beta^\ell \subset Q_\alpha^k \text{ or } Q_\beta^\ell \cap Q_\alpha^k = \emptyset. \tag{3}$$

$$\text{Each } Q_\alpha^k \text{ contains a ball } B(z_\alpha^k, a_0\delta^k). \tag{4}$$

$$\text{Each } Q_\alpha^k \text{ is contained in the ball } B(z_\alpha^k, a_1\delta^k). \tag{5}$$

(David 1988; Christ 1990)

Dyadic cubes on Ahlfors regular spaces

David (1988) showed that Ahlfors regular metric measure spaces contained in Euclidean spaces admit a dyadic cube decomposition. By the Assouad embedding theorem this decomposition holds for Ahlfors regular metric measure spaces.

Christ (1990) gave a construction of a dyadic cube decomposition for the more general case of spaces of homogeneous type.

Theorem 8

Let X be an Ahlfors regular metric measure space of dimension d. Then there exists a family of dyadic cubes as in Definition 7.

(Assouad 1983; David 1988; Christ 1990; David and Semmes 1990; David 1991)

The construction of an equal measure partition

Let (X, ρ, μ) be a connected Ahlfors regular metric measure space of dimension d with constants $0 < c_1 < c_2$ as per Definition 6. In addition, let

$$\left\{Q_\alpha^k : k \in \mathbb{Z}, \alpha \in I_k\right\}$$

be a family of dyadic cubes on X, as per Theorem 8, with properties as per Definition 7.

Assume without loss of generality that $\mu(X) = 1$.

Outline of the construction

We construct a partition of X into N regions of measure $1/N$ as follows.

1. Determine a generation n of large cubes
2. Create an adjacency graph Γ
3. Create a directed spanning tree T from Γ
4. Determine an upper bound M
5. Determine a generation m of small cubes
6. Split leaf nodes into regions
7. Split non-leaf non-root nodes into regions
8. Split the root node into regions

(Feige and Schechtman 2002; L 2007)

Step 1: Determine a generation n of large cubes

For "large enough" N let n be the only integer such that

$$a_0\delta^{n+1} < \left(\frac{2}{c_1 N}\right)^{1/d} \leqslant a_0\delta^n,$$

so that

$$\mu(Q_\alpha^n) \geqslant 2/N$$

for all $\alpha \in I_n$.

Step 2: Create an adjacency graph Γ

Using the cubes of generation n, create a connected graph Γ with a vertex for each index $\alpha \in I_n$ and an edge (α, β) for each pair of centre points z_α^n, z_β^n that satisfy

$$B(z_\alpha^n, a_1\delta^n) \cap B(z_\beta^n, a_1\delta^n) \neq \emptyset.$$

Step 3: Create a directed spanning tree T from Γ

Take any spanning tree S of Γ.

Mark a centre vertex as the root.

Create the directed tree T from S by directing the edges from the leaves towards the root:

$(\alpha, \beta) \in T$ means T has an edge from child α to parent β.

(Riordan 1958; Ore 1962)

Step 4: Determine an upper bound M

Determine M independent of N so that

$$\mu\left(Q^n_\beta \cup \bigcup_{(\alpha,\beta)\in T} Q^n_\alpha\right) < \frac{M}{N}$$

for all α, β in generation n.

Step 5: Determine a generation m of small cubes

Let $m := n + k$, where k is a positive integer independent of N such that

$$\mu(Q^m_\eta) \leqslant \mu\left(B\left(z^m_\eta, a_1\delta^m\right)\right) < \frac{1}{MN}$$

for all η in generation m.

Step 6: Split leaf nodes into regions

For each leaf node β, let $N_\beta := \lfloor N\,\mu(Q^n_\beta)\rfloor$.

This is the maximum number of regions of measure $1/N$ that fit into Q^n_β.

Choose N_β cubes of generation m within Q^n_β to be the nuclei of regions.

Extend each nucleus into a region of measure $1/N$ within Q^n_β.

Let W_β be the rest of Q^n_β.

Step 7: Split non-leaf non-root nodes into regions

For each non-leaf node β other than the root, let

$$X_\beta := Q^n_\beta \cup \bigcup_{(\alpha,\beta)\in T} W_\alpha \, .$$

Let $N_\beta := \lfloor N \, \mu(X_\beta) \rfloor$.

Choose N_β cubes of generation m within Q^n_β to be the nuclei of regions.

Take a subset W_β of Q^n_β, disjoint from these nuclei, of measure $\mu\,(X_\beta) - N_\beta/N$.

Extend each nucleus into a region of measure $1/N$ within $X_\beta \setminus W_\beta$.

Step 8: Split the root node into regions

For the root node γ, define $X_\gamma := Q_\gamma^n \cup \bigcup_{(\beta,\gamma)\in T} W_\beta$.

We have $\mu(X_\gamma) = N_\gamma / N$, where $N_\gamma := N - \sum_{\alpha\neq\gamma} N_\alpha$.

Choose N_γ cubes of generation m within Q_γ^n to be the nuclei of regions.

Extend each nucleus into a region of measure $1/N$ within X_γ.