Logarithmic Equilibrium on the Sphere in the Presence of Multiple Point Charges

Alan Legg

Purdue Fort Wayne

Point Distribution Webinar

This talk will present joint work with P. Dragnev, building on a paper of Brauchart, Dragnev, Saff, Womersley (2018).

This talk will present joint work with P. Dragnev, building on a paper of Brauchart, Dragnev, Saff, Womersley (2018).

- Use the sphere S^2 as the setting for a familiar problem from potential theory:

This talk will present joint work with P. Dragnev, building on a paper of Brauchart, Dragnev, Saff, Womersley (2018).

- Use the sphere S^2 as the setting for a familiar problem from potential theory:
- How does a charge placed on a conductor distribute to obtain a configuration of minimal energy, in the presence of an external field?

This talk will present joint work with P. Dragnev, building on a paper of Brauchart, Dragnev, Saff, Womersley (2018).

- Use the sphere S^2 as the setting for a familiar problem from potential theory:
- How does a charge placed on a conductor distribute to obtain a configuration of minimal energy, in the presence of an external field?
- In the plane, with logarithmic interactions between charges, if a charge is placed onto a domain $\Omega \subset \mathbb{C}$, what is the equilibrium distribution of the charge?

For example, on a bounded smooth finitely connected domain Ω, we expect the charge to repulse itself as far as possible and reside only on the outer boundary.

- Given a positive unit Borel measure μ as a charge distribution, say with compact support in $\bar{\Omega}$, its potential at point z in the plane is:

- Given a positive unit Borel measure μ as a charge distribution, say with compact support in $\bar{\Omega}$, its potential at point z in the plane is:
- $$U^{\mu}(z) = \int \ln \frac{1}{|z-w|} d\mu(w).$$

- Given a positive unit Borel measure μ as a charge distribution, say with compact support in $\bar{\Omega}$, its potential at point z in the plane is:
- $$U^{\mu}(z) = \int \ln \frac{1}{|z-w|} d\mu(w).$$
- The logarithmic energy of the distribution is:

- Given a positive unit Borel measure μ as a charge distribution, say with compact support in $\bar{\Omega}$, its potential at point z in the plane is:
-
$$U^{\mu}(z) = \int \ln \frac{1}{|z-w|} d\mu(w).$$

- The logarithmic energy of the distribution is:

$$I_{\mu} = \int U^{\mu} d\mu = \int \int \ln \frac{1}{|z-w|} d\mu(w) d\mu(z).$$

- Given a positive unit Borel measure μ as a charge distribution, say with compact support in $\bar{\Omega}$, its potential at point z in the plane is:
- $$U^{\mu}(z) = \int \ln \frac{1}{|z-w|} d\mu(w).$$
- The logarithmic energy of the distribution is:
$$I_{\mu} = \int U^{\mu} d\mu = \int \int \ln \frac{1}{|z-w|} d\mu(w) d\mu(z).$$
- The minimal energy problem is to determine which positive unit Borel measure μ supported in $\bar{\Omega}$ will minimize I_{μ}.

When the ‘conductor’ Ω has positive logarithmic capacity:

When the 'conductor' Ω has positive logarithmic capacity:

- The minimal energy $\inf_\mu I_\mu$ is finite and obtained by some μ.

When the 'conductor' Ω has positive logarithmic capacity:

- The minimal energy $\inf_\mu I_\mu$ is finite and obtained by some μ.
- The energy-minimizing measure is unique. Call it μ_E, the equilibrium measure.

- What happens in the presence of an external electric field Q?

- What happens in the presence of an external electric field Q?
- We expect that the external field will influence the equilibrium distribution.

- What happens in the presence of an external electric field Q?
- We expect that the external field will influence the equilibrium distribution.
- In general, the possible equilibrium supports in this case are more diverse.

For the right kinds of external fields (called ‘admissible’), there is a theory of equilibrium measures which has similarities to the theory without external fields.

For the right kinds of external fields (called ‘admissible’), there is a theory of equilibrium measures which has similarities to the theory without external fields. For a conductor Ω in the plane in the presence of an external field Q, place a unit charge on Ω according to a measure μ. The external field influences the energy of the system.

For the right kinds of external fields (called ‘admissible’), there is a theory of equilibrium measures which has similarities to the theory without external fields. For a conductor Ω in the plane in the presence of an external field Q, place a unit charge on Ω according to a measure μ. The external field influences the energy of the system. The *weighted* energy of the measure μ in the presence of Q is:

For the right kinds of external fields (called ‘admissible’), there is a theory of equilibrium measures which has similarities to the theory without external fields. For a conductor Ω in the plane in the presence of an external field Q, place a unit charge on Ω according to a measure μ. The external field influences the energy of the system. The *weighted* energy of the measure μ in the presence of Q is:

$$I_\mu^Q = \int\int \ln \frac{1}{|z-w|} d\mu(z)d\mu(w) + 2\int Q(z)d\mu(z).$$

For the right kinds of external fields (called ‘admissible’), there is a theory of equilibrium measures which has similarities to the theory without external fields. For a conductor Ω in the plane in the presence of an external field Q, place a unit charge on Ω according to a measure μ. The external field influences the energy of the system. The *weighted* energy of the measure μ in the presence of Q is:

$$I_\mu^Q = \int\int \ln \frac{1}{|z-w|} d\mu(z) d\mu(w) + 2\int Q(z) d\mu(z).$$

The equilibrium measure in the presence of Q is then the positive unit Borel measure which will minimize the weighted energy I_μ^Q.

A fundamental tool in this situation is the Frostman Theorem:

A fundamental tool in this situation is the Frostman Theorem:

Theorem

In the external field situation described above, if Ω *has positive logarithmic capacity:*

- *The minimal energy* $\inf_\mu I_\mu^Q = V_Q$ *is finite.*

A fundamental tool in this situation is the Frostman Theorem:

Theorem

In the external field situation described above, if Ω has positive logarithmic capacity:

- *The minimal energy* $\inf_{\mu} I_{\mu}^{Q} = V_Q$ *is finite.*
- *The minimal energy* V_Q *is obtained by a unique measure* μ_Q*.*

A fundamental tool in this situation is the Frostman Theorem:

Theorem

In the external field situation described above, if Ω has positive logarithmic capacity:

- *The minimal energy $\inf_\mu I_\mu^Q = V_Q$ is finite.*
- *The minimal energy V_Q is obtained by a unique measure μ_Q.*
- *For some constant F_Q, $U^{\mu_Q}(z) + Q(z) = F_Q$ on $supp(\mu_Q)$, and $U_{\mu_Q}(z) + Q(z) \geq F_Q$ on all of $\mathbb{C}$ (q.e.).*

A fundamental tool in this situation is the Frostman Theorem:

Theorem

In the external field situation described above, if Ω has positive logarithmic capacity:

- *The minimal energy* $\inf_\mu I_\mu^Q = V_Q$ *is finite.*
- *The minimal energy* V_Q *is obtained by a unique measure* μ_Q.
- *For some constant* F_Q, $U^{\mu_Q}(z) + Q(z) = F_Q$ *on* $supp(\mu_Q)$, *and* $U_{\mu_Q}(z) + Q(z) \geq F_Q$ *on all of* $\mathbb{C}$ *(q.e.).*
- *The measure* μ_Q *is characterized by the previous item.*

A fundamental tool in this situation is the Frostman Theorem:

Theorem

In the external field situation described above, if Ω has positive logarithmic capacity:

- *The minimal energy* $\inf_\mu I_\mu^Q = V_Q$ *is finite.*
- *The minimal energy* V_Q *is obtained by a unique measure* μ_Q*.*
- *For some constant* F_Q*,* $U^{\mu_Q}(z) + Q(z) = F_Q$ *on* $\mathrm{supp}(\mu_Q)$*, and* $U_{\mu_Q}(z) + Q(z) \geq F_Q$ *on all of* $\mathbb{C}$ *(q.e.).*
- *The measure* μ_Q *is characterized by the previous item.*

Note: the equilibrium measure has constant 'weighted potential' on its support. This is intuitively appealing, since if there were a potential difference, a current would flow to equalize it.

Now consider the same problem on the sphere S^2.

Now consider the same problem on the sphere S^2.

- If the sphere is a conductor,

Now consider the same problem on the sphere S^2.

- If the sphere is a conductor,
- and we place a unit positive charge on it

Now consider the same problem on the sphere S^2.

- If the sphere is a conductor,
- and we place a unit positive charge on it
- which is free to redistribute in the presence of an external field

Now consider the same problem on the sphere S^2.

- If the sphere is a conductor,
- and we place a unit positive charge on it
- which is free to redistribute in the presence of an external field
- what is the equilibrium given an external field Q?

In this case, we can recover the planar problem via stereographic projection: surface measure on the sphere corresponds to the measure $\frac{dA}{(1+|z|^2)^2}$ in the plane, when the north pole of the sphere is mapped to ∞.

Brauchart, Dragnev, Saff, and Womersley were able to determine what happens when the external field consists of finitely many point charges that are sufficiently weak.

Their result is the following (paraphrasing):

Theorem

(BDSW) Let a unit charge be placed uniformly on the sphere, and then place finitely many sufficiently separated point charges on the sphere (with logarithmic interactions). The energy-minimizing charge distribution in the presence of these point charges is still uniform, but the support excludes perfect spherical caps centered on the point charges. The size of the caps can be explicitly calculated.

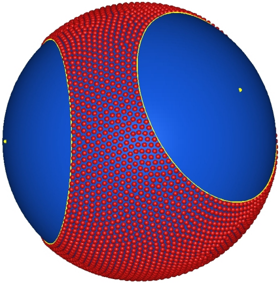
Their result is the following (paraphrasing):

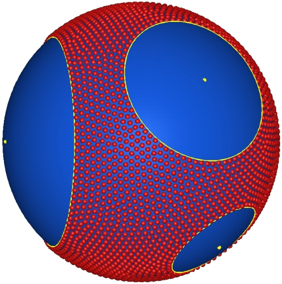
Theorem

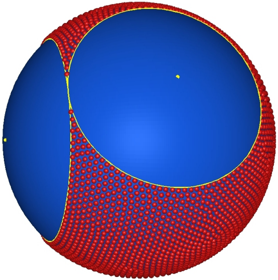
(BDSW) Let a unit charge be placed uniformly on the sphere, and then place finitely many sufficiently separated point charges on the sphere (with logarithmic interactions). The energy-minimizing charge distribution in the presence of these point charges is still uniform, but the support excludes perfect spherical caps centered on the point charges. The size of the caps can be explicitly calculated.

In other words, each point charge has a 'cap of influence' where it tends to repulse the charge on the sphere. 'Sufficiently weak' means that the caps should be disjoint.

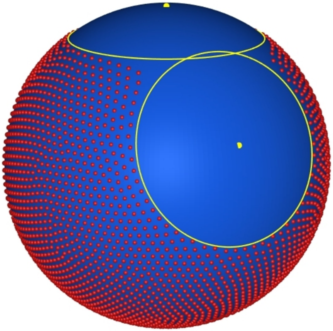
Images from Brauchart, Dragnev, Saff, and Womersely: Logarithmic and Riesz Equilibrium for Multiple Sources on the Sphere: The Exceptional Case, Contemporary Computational Mathematics-A celebration of the 80th Birthday of Ian Sloan, 179–203.



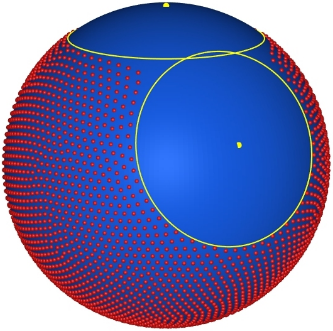




My interest in the problem comes from a question raised in their paper:



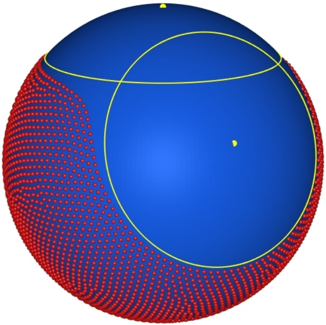
My interest in the problem comes from a question raised in their paper:



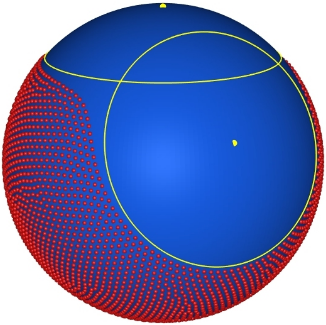
What's going on here?

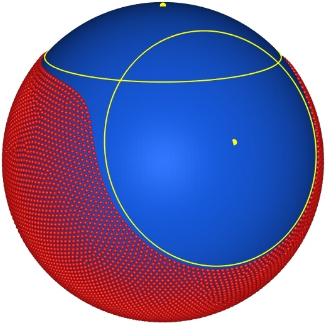
A couple other views:

A couple other views:

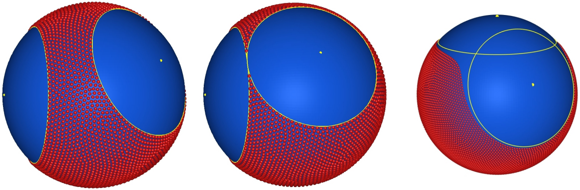


A couple other views:





It appears that the moment after caps of influence overlap, the equilibrium support smooths out into a lobed shape. Can we describe the shape?

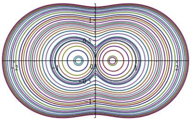


In the plane, when two circles combine in a potential theory setting and smooth out into a single curve, it makes one think of a Neumann Oval, which is a type of Quadrature Domain.

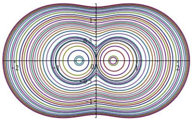
- ‘Neumann Ovals’ are the way to overlap two circles in the sense of the harmonic mean value theorem.

- 'Neumann Ovals' are the way to overlap two circles in the sense of the harmonic mean value theorem.
- On a Neumann Oval, the integral of a harmonic function is the linear combination of function values at 2 nodes, just as in a disc the integral is a multiple of the function value at the center.

- ‘Neumann Ovals’ are the way to overlap two circles in the sense of the harmonic mean value theorem.
- On a Neumann Oval, the integral of a harmonic function is the linear combination of function values at 2 nodes, just as in a disc the integral is a multiple of the function value at the center.
- As the coefficients at each node in the linear combination increase, the region grows from a union of 2 discs, into a lobed shape, and finally into a convex shape.



- ‘Neumann Ovals’ are the way to overlap two circles in the sense of the harmonic mean value theorem.
- On a Neumann Oval, the integral of a harmonic function is the linear combination of function values at 2 nodes, just as in a disc the integral is a multiple of the function value at the center.
- As the coefficients at each node in the linear combination increase, the region grows from a union of 2 discs, into a lobed shape, and finally into a convex shape.



- There are corresponding shapes for any number of nodes.

A Quadrature Domain is a domain Ω in the plane where integrating functions in a given test class (usually, harmonic L^1, Bergman Space A^2, integrable analytic AL^1, etc.) coincides with taking a linear combination of point evaluations of the functions and their derivatives. The same coefficients and points should work for any function in the test class:

$$\int_\Omega h(z)dA = \sum_{i,j} c_{i,j} h^{(j)}(z_i).$$

The number of terms in the sum is the 'order' of the Q.D., and the points of evaluation z_i are called the 'nodes.'

Examples of Q.D.'s:
Disc (the only Q.D. of order 1)
Neumann Oval (order 2 with distinct nodes)
Cardioid/Limacon (order 2 with single node)
Several overlapped discs merged together in the right way

These domains are very appealing in complex analysis, because they automatically enjoy many strong properties.

- Algebraic Boundary
- Algebraic proper maps to the unit disc
- Maps between Quadrature Domains are algebraic
- Algebraic Bergman Kernel
- Meromorphic Schwarz Function

- In fact, any bounded simply connected Q.D. is a rational image of the unit disc. So a conformal map from the disc to a Neumann Oval is of the form

$$f(z) = \frac{A}{z - B} + \frac{C}{z - D}$$

- In fact, any bounded simply connected Q.D. is a rational image of the unit disc. So a conformal map from the disc to a Neumann Oval is of the form

$$f(z) = \frac{A}{z - B} + \frac{C}{z - D}$$

- Similarly for higher numbers of nodes: the boundary of a Q.D. is always algebraic.

About the ‘Schwarz Function:’

About the ‘Schwarz Function:’

- Given a bounded real analytic curve in the plane, there is a neighborhood of the curve where the function $\bar{z}$ extends to be analytic.

About the ‘Schwarz Function:’

- Given a bounded real analytic curve in the plane, there is a neighborhood of the curve where the function $\bar{z}$ extends to be analytic.
- This continuation is called the Schwarz Function $S(z)$ when the curve is the boundary of a domain Ω.

About the ‘Schwarz Function:’

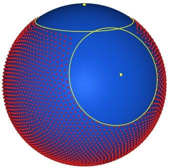
- Given a bounded real analytic curve in the plane, there is a neighborhood of the curve where the function $\bar{z}$ extends to be analytic.
- This continuation is called the Schwarz Function $S(z)$ when the curve is the boundary of a domain Ω.
- To be a quadrature domain means that the Schwarz function extends meromorphically throughout Ω with finitely many poles. The poles are the quadrature nodes.

- For example, on the unit circle, $\bar{z} = \frac{1}{z}$, which extends meromorphically all the way inside the disc and has a pole at 0, which is the node of evaluation for the harmonic mean value theorem.

- For example, on the unit circle, $\bar{z} = \frac{1}{z}$, which extends meromorphically all the way inside the disc and has a pole at 0, which is the node of evaluation for the harmonic mean value theorem.
- In general, if the Schwarz function of Ω is meromorphic and $h(z)$ is integrable and analytic, $\int_\Omega h(z) dz \wedge d\bar{z} = \int_\Omega d(h(z)\bar{z} dz)$.

- For example, on the unit circle, $\bar{z} = \frac{1}{z}$, which extends meromorphically all the way inside the disc and has a pole at 0, which is the node of evaluation for the harmonic mean value theorem.
- In general, if the Schwarz function of Ω is meromorphic and $h(z)$ is integrable and analytic, $\int_\Omega h(z)dz \wedge d\bar{z} = \int_\Omega d(h(z)\bar{z}dz)$.
- So by Stokes's Theorem, $\int_\Omega hdz \wedge d\bar{z} = \int_{\partial\Omega} \bar{z}h(z)dz = \int_{\partial\Omega} h(z)S(z)dz$. Since S is meromorphic, this just becomes a linear combination of evaluations of $h(z)$ and its derivatives by the Residue Theorem.

Back to the problem at hand:



This region of charge exclusion looks like a Neumann Oval (Q.D.) which was stereographically projected onto S^2. And it was made in a similar way, by overlapping two circles in a potential-theoretic way.

That turns out to be a good inspiration. In fact, we can prove that the region of exclusion is the projection of a Q.D., using classical Complex Analysis after a stereographic projection.

Assume for now that the complement of the equilibrium support is smooth and simply connected. Rotate the sphere so that the north pole is in the equilibrium support. Then project stereographically to the plane. As described in the BDSW paper, we can write a planar formulation of the Frostman condition of constant weighted potential of the equilibrium measure.

Let $\Omega \subset \mathbb{C}$ be the stereographically projected support of the equilibrium measure (unbounded by our choice of north pole), and assume its complement (the projected region of charge exclusion) is smooth and simply connected. The Frostman Theorem condition will involve a potential, the external field from the 2 point charges, and some pieces arising from the distortion caused by the stereographic projection.

Let $\Omega \subset \mathbb{C}$ be the stereographically projected support of the equilibrium measure (unbounded by our choice of north pole), and assume its complement (the projected region of charge exclusion) is smooth and simply connected. The Frostman Theorem condition will involve a potential, the external field from the 2 point charges, and some pieces arising from the distortion caused by the stereographic projection.

- Let the point charges be q_1, q_2, and let their planar projections be located at $z_1, z_2 \in \mathbb{C}$, $Q = q_1 + q_2$. Ω will satisfy:

Let $\Omega \subset \mathbb{C}$ be the stereographically projected support of the equilibrium measure (unbounded by our choice of north pole), and assume its complement (the projected region of charge exclusion) is smooth and simply connected. The Frostman Theorem condition will involve a potential, the external field from the 2 point charges, and some pieces arising from the distortion caused by the stereographic projection.

- Let the point charges be q_1, q_2, and let their planar projections be located at $z_1, z_2 \in \mathbb{C}$, $Q = q_1 + q_2$. Ω will satisfy:
-

$$\frac{Q+1}{\pi} \int_{\Omega} \frac{1}{(1+|w|^2)^2} \ln \frac{1}{|z-w|} dA_w + q_1 \ln \frac{1}{|z-z_1|} + q_2 \ln \frac{1}{|z-z_2|}$$

$$+(Q+1) \ln \sqrt{1+|z|^2} = const,$$

valid for $z \in \Omega$.

Let $\Omega \subset \mathbb{C}$ be the stereographically projected support of the equilibrium measure (unbounded by our choice of north pole), and assume its complement (the projected region of charge exclusion) is smooth and simply connected. The Frostman Theorem condition will involve a potential, the external field from the 2 point charges, and some pieces arising from the distortion caused by the stereographic projection.

- Let the point charges be q_1, q_2, and let their planar projections be located at $z_1, z_2 \in \mathbb{C}$, $Q = q_1 + q_2$. Ω will satisfy:
- $$\frac{Q+1}{\pi}\int_{\Omega}\frac{1}{(1+|w|^2)^2}\ln\frac{1}{|z-w|}dA_w + q_1\ln\frac{1}{|z-z_1|} + q_2\ln\frac{1}{|z-z_2|}$$
  $$+(Q+1)\ln\sqrt{1+|z|^2} = const,$$
  valid for $z \in \Omega$.
- Note: this projected version does not involve an admissible external field-but it does involve a 'weakly admissible' external field.

- Consider the integral as occurring within a large radius that goes to ∞.

- Consider the integral as occurring within a large radius that goes to ∞.
- Differentiate in z, and use Green's Theorem in the form of the C^∞ version of the Cauchy Integral Formula: for a function ϕ smooth up to the boundary of a bounded smooth domain D,

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\phi(w)}{w-z}dw + \frac{1}{2\pi i}\int\int_D \frac{d\varphi/d\bar{w}}{w-z}dA_w.$$

- Consider the integral as occurring within a large radius that goes to ∞.
- Differentiate in z, and use Green's Theorem in the form of the C^∞ version of the Cauchy Integral Formula: for a function ϕ smooth up to the boundary of a bounded smooth domain D,

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\phi(w)}{w-z}dw + \frac{1}{2\pi i}\int\int_D \frac{d\varphi/d\bar{w}}{w-z}dA_w.$$

- After some algebra, and noticing that some terms vanish as the radius goes to ∞, you'll get:

- Consider the integral as occurring within a large radius that goes to ∞.
- Differentiate in z, and use Green's Theorem in the form of the C^∞ version of the Cauchy Integral Formula: for a function ϕ smooth up to the boundary of a bounded smooth domain D,

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\phi(w)}{w-z}dw + \frac{1}{2\pi i}\int\int_D \frac{d\varphi/d\bar{w}}{w-z}dA_w.$$

- After some algebra, and noticing that some terms vanish as the radius goes to ∞, you'll get:
- $\frac{1}{2\pi i}\int_{\partial\Omega^c}\frac{1}{w+\frac{1}{\bar{w}}}\frac{1}{w-z}dw = \frac{q_1}{Q+1}\frac{1}{z_1-z} + \frac{q_2}{Q+1}\frac{1}{z_2-z}.$

- This formula itself can be differentiated in z however many times we want:
$\frac{1}{2\pi i}\int_{\partial\Omega^c}\frac{1}{w+\frac{1}{\bar{w}}}\frac{1}{(w-z)^m}dw = \frac{q_1}{Q+1}\frac{1}{(z_1-z)^m}+\frac{q_2}{Q+1}\frac{1}{(z_2-z)^m}$.
This holds for positive integers m.

- This formula itself can be differentiated in z however many times we want:
$\frac{1}{2\pi i}\int_{\partial\Omega^c}\frac{1}{w+\frac{1}{\bar{w}}}\frac{1}{(w-z)^m}dw = \frac{q_1}{Q+1}\frac{1}{(z_1-z)^m} + \frac{q_2}{Q+1}\frac{1}{(z_2-z)^m}.$
This holds for positive integers m.
- But now by linearity/partial fractions, notice that this amounts to saying that for any rational function R with poles in Ω:

- $$\frac{1}{2\pi i}\int_{\partial\Omega^c}\frac{1}{w+\frac{1}{\bar{w}}}R(w)dw = \frac{q_1}{Q+1}R(z_1) + \frac{q_2}{Q+1}R(z_2).$$

  That's starting to look like a quadrature domain with 2 nodes!

- $$\frac{1}{2\pi i}\int_{\partial\Omega^c}\frac{1}{w+\frac{1}{\bar{w}}}R(w)dw = \frac{q_1}{Q+1}R(z_1) + \frac{q_2}{Q+1}R(z_2).$$

  That's starting to look like a quadrature domain with 2 nodes!
- To confirm our suspicion, use Mergelyan's Theorem. The rational R in the above equality can be replaced by uniform approximation to be any $h \in A^\infty(\Omega^c)$. Then write the function values on the right side in terms of the Cauchy Integral Formula, rearrange and get:

- $$\frac{1}{2\pi i}\int_{\partial\Omega^c}\frac{1}{w+\frac{1}{\bar{w}}}R(w)dw = \frac{q_1}{Q+1}R(z_1) + \frac{q_2}{Q+1}R(z_2).$$

  That's starting to look like a quadrature domain with 2 nodes!
- To confirm our suspicion, use Mergelyan's Theorem. The rational R in the above equality can be replaced by uniform approximation to be any $h \in A^\infty(\Omega^c)$. Then write the function values on the right side in terms of the Cauchy Integral Formula, rearrange and get:
- $$\int_{\partial\Omega^c}\left(\frac{1}{w+\frac{1}{\bar{w}}} - \frac{1}{w-z_1} - \frac{1}{w-z_2}\right)h(w)dw = 0.$$

By the structure of the orthogonal complement of the Hardy Space on a smooth bounded domain,there is a function $H \in A^{\infty}(\Omega^c)$ giving on the boundary:

By the structure of the orthogonal complement of the Hardy Space on a smooth bounded domain,there is a function $H \in A^\infty(\Omega^c)$ giving on the boundary:
$\frac{1}{w+\frac{1}{\bar{w}}} - \frac{1}{w-z_1} - \frac{1}{w-z_2} = H(w)$,
for any $w \in \partial\Omega^c$.

By the structure of the orthogonal complement of the Hardy Space on a smooth bounded domain,there is a function $H \in A^{\infty}(\Omega^c)$ giving on the boundary:

$\frac{1}{w+\frac{1}{\bar{w}}} - \frac{1}{w-z_1} - \frac{1}{w-z_2} = H(w),$

for any $w \in \partial\Omega^c$.

Now solve for $\bar{w}$: it has the boundary values of a meromorphic function. In other words, Ω^c is a Quadrature Domain!

Using the Argument Principle, you can conclude that the Schwarz function $S(w)$ has exactly 2 poles, which means Ω^c is a Quadrature Domain of order 2.

Using the Argument Principle, you can conclude that the Schwarz function $S(w)$ has exactly 2 poles, which means Ω^c is a Quadrature Domain of order 2.

For example, you can tell $\frac{1}{w+\frac{1}{S}}$ has exactly 2 poles, which means $w + \frac{1}{S}$ has exactly 2 roots. On the other hand, $w + \frac{1}{\bar{w}} = \frac{1+|w|^2}{\bar{w}}$, which has the same winding number as w (numerator has argument 0, conjugating negates, and so does being in the denominator). Using $zeroes - poles = winding$, you can rig the arithmetic to give you that S has 2 poles.

Using the Argument Principle, you can conclude that the Schwarz function $S(w)$ has exactly 2 poles, which means Ω^c is a Quadrature Domain of order 2.

For example, you can tell $\frac{1}{w+\frac{1}{S}}$ has exactly 2 poles, which means $w+\frac{1}{S}$ has exactly 2 roots. On the other hand, $w+\frac{1}{\bar{w}}=\frac{1+|w|^2}{\bar{w}}$, which has the same winding number as w (numerator has argument 0, conjugating negates, and so does being in the denominator). Using $zeroes - poles = winding$, you can rig the arithmetic to give you that S has 2 poles.

In other words, our initial suspicion was correct-the region of charge exclusion on the sphere is the stereographic preimage of a quadrature domain of order 2: a Neumann Oval.

With some extra bookkeeping in the calculations, the same idea will extend to any finite number of charges. A set of n point charges will result in a region of charge exclusion which is the stereographic preimage of a Q.D. with n nodes in the plane.

- The approach we just saw is reliant on enough smoothness to use Stokes' Theorem, and becomes awkward when the complement of the equilibrium support is disconnected.

- The approach we just saw is reliant on enough smoothness to use Stokes' Theorem, and becomes awkward when the complement of the equilibrium support is disconnected.
- The connectedness can be handled by balayage: the problem can be solved for a single component, because disconnected regions of charge exclusion do not interact.
- There is an alternate approach which gives the result without assumptions on boundary smoothness.

- The approach we just saw is reliant on enough smoothness to use Stokes' Theorem, and becomes awkward when the complement of the equilibrium support is disconnected.
- The connectedness can be handled by balayage: the problem can be solved for a single component, because disconnected regions of charge exclusion do not interact.
- There is an alternate approach which gives the result without assumptions on boundary smoothness.
- It turns out that the density of the equilibrium measure is always the Laplacian of the external field, throughout the equilibrium support regardless of boundary (Hedenmalm-Makarov). This allows analyzing the Frostman Condition directly.

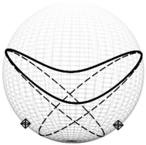
- The approach we just saw is reliant on enough smoothness to use Stokes' Theorem, and becomes awkward when the complement of the equilibrium support is disconnected.
- The connectedness can be handled by balayage: the problem can be solved for a single component, because disconnected regions of charge exclusion do not interact.
- There is an alternate approach which gives the result without assumptions on boundary smoothness.
- It turns out that the density of the equilibrium measure is always the Laplacian of the external field, throughout the equilibrium support regardless of boundary (Hedenmalm-Makarov). This allows analyzing the Frostman Condition directly.
- This yields boundary information in a “weakly holomorphic” sense, which by Weyl's Lemma is legitimately holomorphic. In particular, a meromorphic Schwarz function arises, leading again to a quadrature domain.

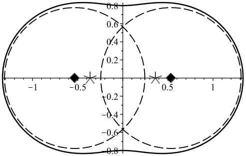
- Kuijlaars and Criado del Rey have independently studied solutions to the present problem in the case of two symmetric charges, and the case of several symmetrically positioned equal charges.

- Kuijlaars and Criado del Rey have independently studied solutions to the present problem in the case of two symmetric charges, and the case of several symmetrically positioned equal charges.
- Via conformal mapping, the quadrature domain approach we just saw obtains exact solutions in the case of two equal charges. In principle, the case of two asymmetric charges can be thought of this way also (the region of charge exclusion is either 2 discs or simply connected). The relationship between the conformal mapping parameters, quadrature data in the plane and charge locations on the sphere turns out to be algebraic, but unpleasant.

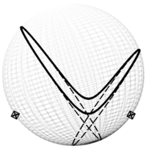
- Kuijlaars and Criado del Rey have independently studied solutions to the present problem in the case of two symmetric charges, and the case of several symmetrically positioned equal charges.
- Via conformal mapping, the quadrature domain approach we just saw obtains exact solutions in the case of two equal charges. In principle, the case of two asymmetric charges can be thought of this way also (the region of charge exclusion is either 2 discs or simply connected). The relationship between the conformal mapping parameters, quadrature data in the plane and charge locations on the sphere turns out to be algebraic, but unpleasant.
- A useful tool for analysis in all cases is the "Spherical Schwarz Function" $S(z)/(1 + zS(z))$, which incorporates the stereographically projected spherical area measure into the usual Schwarz function $S(z)$.

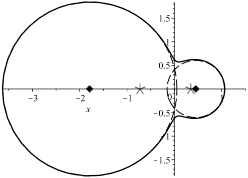
Two symmetric point charges of intensity $q = \frac{41-3\sqrt{41}}{82}$ at the points $(\pm\frac{16}{25}, 0, -\frac{3\sqrt{41}}{25})$ of the unit sphere.



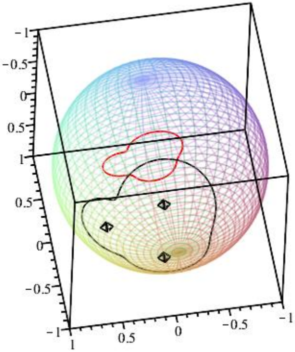


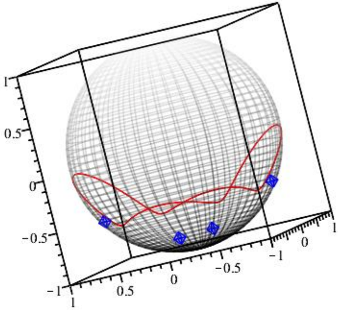
Two asymmetric point charges placed at $(-0.95, 0, -0.31)$ and $(0.62, 0, -0.79)$ with respective intensities. $q_1 = 0.12$, $q_2 = 0.07$.





3 and 4 point configurations





Further questions:

1) Does the equilibrium support itself (as opposed to the complement) have a generalized 'quadrature' structure (called a "mother body"?) This question arises from Kuijlaars and Criado del Rey's approach.

2) What happens on other manifolds?

3) What happens for other potentials, such as Riesz potentials of different exponent?

4) What happens in higher dimension? It is known that quadrature domains are not as well behaved in higher dimensions, and the connection to complex analysis likely breaks down.

Thanks!