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The spherical ensemble with external sources

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Point Distributions Webinar, 10 February 2021

1 Outline

1 Introduction: The spherical ensemble

n random points on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with joint probability density

$$\frac{1}{Z_n} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \|x_i - x_j\|^2$$

- **Determinantal point process**
- **Eigenvalues of a random matrix** **Krishnapur (2009)**

  Take independent Ginibre matrices A and B.

  Compute eigenvalues of AB^{-1}.

  Put points on $\mathbb{S}^2$ with inverse stereographic projection.

1 The spherical ensemble with external charges

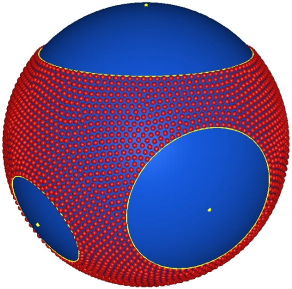
Fix $p_0, \ldots, p_r \in \mathbb{S}^2$ and $a_0, \ldots, a_r > 0$. It creates probability density for points $x_1, \ldots, x_n$ on the sphere

$$\frac{1}{Z_n} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \|x_i - x_j\|^2 \cdot \prod_{j=1}^{n} \prod_{k=0}^{r} \|x_j - p_k\|^{2na_k}$$

Each p_k creates a region around itself that is free from x_j's in large n limit.

When a_k's are small these regions are spherical caps

Brauchart, Dragnev, Saff, Womersley



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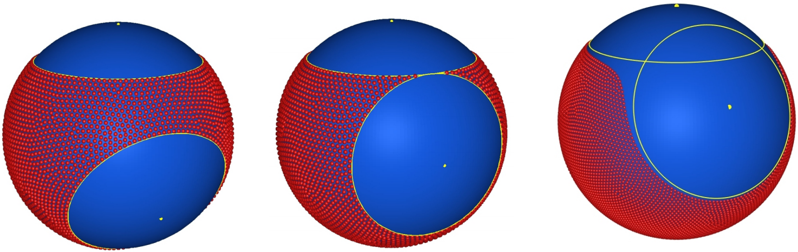
1 The spherical ensemble with external charges

When charges a_k are larger the regions start to overlap.



Figures from BDSW

- **What region D is occupied by the free charges?**
- **D is known as the droplet.**

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1 Normal matrix model

Eigenvalues of complex Ginibre matrix have joint density

$$\frac{1}{Z_n} \prod_{i=1}^{n} \prod_{j=i+1}^{n} |z_i - z_j|^2 \prod_{j=1}^{n} e^{-n|z_j|^2}$$

- **Eigenvalues fill out a disk as $n \to \infty$.**

Ginibre model with external sources (also arises as eigenvalues in a normal matrix model)

$$\frac{1}{Z_n} \prod_{i=1}^{n} \prod_{j=i+1}^{n} |z_i - z_j|^2 \prod_{j=1}^{n} e^{-n|z_j|^2} \prod_{j=1}^{n} \prod_{k=0}^{r} |z_j - q_k|^{2na_k}$$

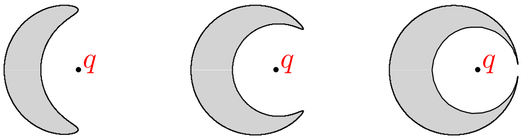
- **Studied in detail for case $r = 0$ by**

Balogh, Bertola, Lee, McLaughlin 2015

1 Droplet

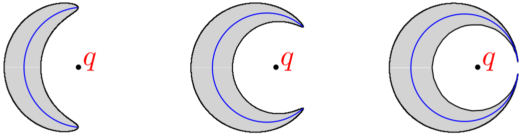
$$\frac{1}{Z_n} \prod_{i=1}^{n} \prod_{j=i+1}^{n} |z_i - z_j|^2 \prod_{j=1}^{n} e^{-n|z_j|^2} \prod_{j=1}^{n} |z_j - q|^{2na}$$

Change in droplet as a increases.



1 Motherbody

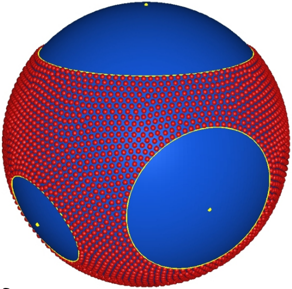
Average characteristic polynomial $P_n(z) = \mathbb{E}\left[\prod_{j=1}^{n}(z - z_j)\right]$ **is planar orthogonal polynomial with zeros that cluster around a contour as $n \to \infty$, known as the motherbody**



- **Planar orthogonality can be rewritten as orthogonality along a closed contour around q.**
  Balogh, Bertola, Lee, McLaughlin 2015, Lee, Yang 2019
- **This allows a Riemann-Hilbert asymptotic analysis**

2 Outline

2 Potential theory



Back to spherical model

$$\frac{1}{Z_n}\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\|x_i-x_j\|^2\cdot\prod_{j=1}^{n}\prod_{k=0}^{r}\|x_j-p_k\|^{2}$$

Fixed charge distribution $\sigma = \sum_{k=0}^{r} a_k \delta_{p_k}$

Limiting distribution μ_σ **of free charges is unique probability measure on** $\mathbb{S}^2$ **that satisfies**

$$U^{\mu_\sigma} + U^{\sigma} = \ell \quad \textbf{on } D = \mathrm{supp}(\mu_\sigma),$$
$$U^{\mu_\sigma} + U^{\sigma} \geq \ell \quad \textbf{on } \mathbb{S}^2.$$

Notation: $U^{\mu}(x) = \int \log \frac{1}{\| \cdots \|} d\mu(y)$

2 Potential theory

Fixed charge distribution $\sigma = \sum_{k=0}^{r} a_k \delta_{p_k}$

Limiting distribution μ_σ **of free charges is unique probability measure on** $\mathbb{S}^2$ **that satisfies**

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Lemma

$$\boxed{\mu_\sigma = (\lambda(D))^{-1} \lambda_D}$$

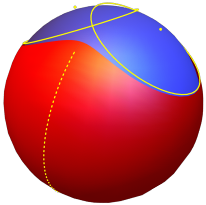
where λ_D ***is the restriction to*** D ***of the normalized Lebesgue measure*** λ ***on*** $\mathbb{S}^2$***.***

2 The motherbody

A motherbody for D is a probability measure σ^* on a one-dimensional subset of D such that

$$U^{\sigma^*} = (\lambda(D))^{-1} U^{\lambda_D} + \ell^* \quad \text{on } \mathbb{S}^2 \setminus D,$$
$$U^{\sigma^*} \geq (\lambda(D))^{-1} U^{\lambda_D} + \ell^* \quad \text{on } D.$$

In case of two points p_0, p_1 with sufficiently large equal charges $a_0 = a_1$, $\mathbb{S}^2 \setminus D$ is an ellipse after stereographic projection onto the complex plane.



The motherbody is minimizer of equilibrium problem with external field
Criado del Rey - K

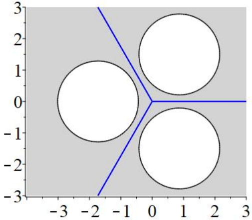
2 More external charges

We generalize this to $r+1$ points $p_0, \ldots, p_r$ with symmetry

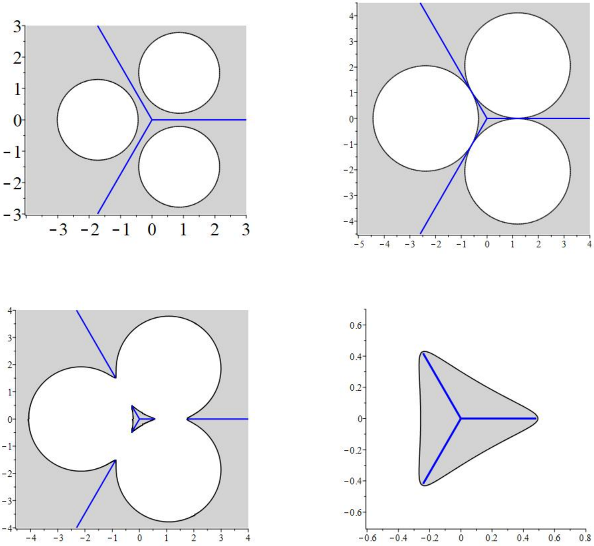
- p_j's are located symmetrically around the north pole
- Equal charges $a_0 = a_1 = \cdots = a_r = a > 0$

Motherbody will be supported on $r+1$ meridians on the sphere, connecting the north and south poles.

After stereographic projection these are $r+1$ half-rays in the complex plane.



2 More pictures for $r = 2$ (three charges)



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3 Outline

1. Introduction
2. Potential theory
3. **Vector equilibrium problem**
4. About the proof

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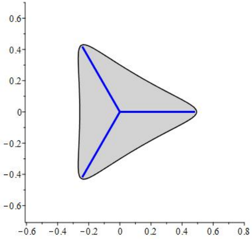
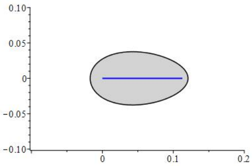
3 Remove the symmetry



$$z \longmapsto z^{r+1}$$



We expect three cases for the support Σ of the motherbody (after removing the symmetry)

- **Bounded interval support (BIS):** $\Sigma = [0, x_1]$
- **Two interval support (TIS):** $\Sigma = [0, x_1] \cup [x_2, \infty)$
- **Full interval support (FIS):** $\Sigma = [0, \infty)$

3 Vector equilibrium problem with r measures

Energy functional with input $q > 0$, $0 < t < 1$ and integer $r \geq 1$

$$\mathcal{E}(\mu_1, \ldots, \mu_r) = \sum_{j=1}^{r} I(\mu_j) - \sum_{j=1}^{r-1} I(\mu_j, \mu_{j+1})$$

$$- \frac{1-t}{t} \int \log \left| x + q^{-1} \right| d\mu_1(x) + \frac{r+t}{t} \int \log |x - (-1)^r q| \, d\mu_r(x)$$

Notation:

$$I(\mu, \nu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\nu(y)$$

$$I(\mu) = I(\mu, \mu)$$

3 Vector equilibrium problem with r measures

Energy functional with input $q > 0$, $0 < t < 1$ and integer $r \geq 1$

$$\mathcal{E}(\mu_1, \ldots, \mu_r) = \sum_{j=1}^{r} I(\mu_j) - \sum_{j=1}^{r-1} I(\mu_j, \mu_{j+1})$$

$$- \frac{1-t}{t} \int \log \left| x + q^{-1} \right| d\mu_1(x) + \frac{r+t}{t} \int \log |x - (-1)^r q| \, d\mu_r(x)$$

Minimize $\mathcal{E}(\mu_1, \ldots, \mu_r)$ under conditions

- μ_j is supported on $[0, \infty)$ if j is odd, and on $(-\infty, 0]$ if j is even,
- $\boxed{\int d\mu_j = 1 + \frac{j-1}{t}}$ for $j = 1, \ldots, r$.

3 Main result

Theorem

(a) ***The vector equilibrium problem has a unique minimizer.***

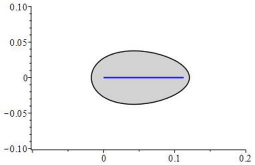
(b) ***The first component*** μ_1 ***has support*** Σ_1 ***that is either*** $[0, x_1]$***, or*** $[0, x_1] \cup [x_2, \infty)$ ***or*** $[0, \infty)$***.***

(c) $\Phi(z) = t \int \frac{d\mu_1(s)}{z - s} + \frac{1-t}{z + q^{-1}}$ ***is an algebraic function with meromorphic continuation to an*** $r + 1$ ***sheeted Riemann surface***

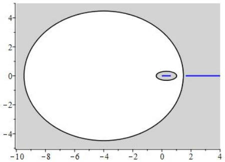
(d) ***There is a domain*** U ***containing*** Σ_1 ***with boundary***

$$\partial U : \quad z\Phi(z) = \frac{|z|^{\frac{2}{r+1}}}{1 + |z|^{\frac{2}{r+1}}}$$

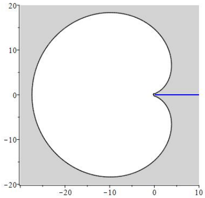
3 Pictures (on different scales)



BIS: closed contour



TIS: two contours



FIS: closed contour

∂U **is boundary of shaded region** U

- **BIS**: $\Sigma_1 = [0, x_1]$ **and** U **is bounded**
- **TIS**: $\Sigma_1 = [0, x_1] \cup [x_2, \infty)$ **and** U **has one bounded and one unbounded component**
- **FIS**: $\Sigma_1 = [0, \infty)$ **and** U **is unbounded**

3 Main result (cont.)

Theorem

(e) $\Omega = \{z \in \mathbb{C} \mid z^{r+1} \in U\}$ ***is mapped by inverse stereographic projection to the droplet*** D

The points $p_0, \ldots, p_r$ ***correspond to equidistant points in the plane on a circle with radius*** $q^{-\frac{1}{r+1}} > 1$

The charges are

$$a = \frac{1-t}{t(r+1)} > 0$$

(f) ***The measure*** μ_1 ***gives the motherbody of*** D ***(after re-introducing the*** $r+1$***-fold symmetry and mapping back to the unit sphere)***

4 Outline

4 Vector equilibrium problem

- **Vector equilibrium problem is weakly admissible**
- **The measures $\mu_2, \ldots, \mu_r$ have full supports $(-\infty, 0]$ or $[0, \infty)$ as they are balayage measures**

$$2\mu_k = \mathrm{Bal}(\mu_{k-1} + \mu_{k+1}; (-1)^{k+1}[0, \infty)), \qquad \textbf{for } k = 2, 3, \ldots, r$$

4 Vector equilibrium problem

- The measures $\mu_2, \ldots, \mu_r$ have full supports $(-\infty, 0]$ or $[0, \infty)$ as they are balayage measures
- μ_1 also minimizes

$$\frac{1}{2} \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \frac{1}{2} \iint \log \frac{1}{|x^{1/r} - y^{1/r}|} d\mu(x) d\mu(y)$$
$$+ \int \left(\left(1 - \tfrac{1}{t}\right) \log \left(x + q^{-1}\right) + \left(1 + \tfrac{r}{t}\right) \log \left(x^{1/r} + q^{1/r}\right) \right) d\mu(x)$$

among probability measures μ on $[0, \infty)$

Equilibrium problem for Muttalib-Borodin ensemble

Claeys Romano 2014, K 2016, Molag 2020

4 The support of μ_1

μ_1 is the balayage measure $\quad \mathrm{Bal}\left(-(\frac{1}{t}-1)\delta_{-q}+\mu_2;[0,\infty)\right)$
provided that this is positive

- This happens in FIS (full interval support) case.

There are four possibilities for the support Σ_1 of μ_1

- $\Sigma_1 = [0, x_1]$ with $0 < x_1 < \infty$ **BIS**
- $\Sigma_1 = [0, x_1] \cup [x_2, \infty)$ with $0 < x_1 < x_2 < \infty$ **TIS**
- $\Sigma_1 = [0, \infty)$ **FIS**
- $\Sigma_1 = [x_2, \infty)$ with $0 < x_2 < \infty$ **UIS**

UIS (unbounded interval support) does not happen for $0 < q \leq 1$, BIS does not happen for $q \geq 1$.

This is proved with iterated balayage algorithm

Dragnev Kuijlaars 1999

4 Monotonicity properties

$\mu_1 = \mu_{1,t}$ **and** $\Sigma_1 = \Sigma_{1,t}$ **vary with** $t \in (0,1)$ **Fix** $0 < q < 1$

Theorem

(a) $\Sigma_{1,t}$ ***increases with*** t**.**

(b) $t\mu_{1,t}$ ***increases with*** t**.**

(c) ***There are two critical values*** $0 < t_{1,cr} < t_{2,cr} < 1$ ***such that***
- ***BIS in case*** $0 < t \leq t_{1,cr}$
- ***TIS in case*** $t_{1,cr} < t < t_{2,cr}$
- ***FIS in case*** $t_{2,cr} \leq t < 1$

(a) is implied by (b) and (b) follows from further analysis of iterated balayage algorithm (+some other tricks)

(c) follows from explicit calculations in BIS and FIS cases.

4 Riemann surface

Riemann surface $\mathcal{R}$ with $r+1$ sheets **($\Sigma_j = \operatorname{supp}(\mu_j)$)**

$$\begin{aligned} \mathcal{R}^{(1)} &= \mathbb{C} \setminus \Sigma_1, \\ \mathcal{R}^{(j)} &= \mathbb{C} \setminus (\Sigma_{j-1} \cup \Sigma_j) \quad \textbf{for } j = 2, \ldots, r, \\ \mathcal{R}^{(r+1)} &= \mathbb{C} \setminus \Sigma_r. \end{aligned}$$

- **Meromorphic function Φ is given on jth sheet by**

$$\Phi^{(j)}(z) = t \int \frac{d\mu_j(s)}{z-s} - t \int \frac{d\mu_{j-1}(s)}{z-s}, \qquad z \in \mathcal{R}^{(j)}$$

with $\mu_0 = \left(1 - \frac{1}{t}\right) \delta_{-q^{-1}}$ and $\mu_{r+1} = \left(1 + \frac{r}{t}\right) \delta_{(-1)^r q}$

- **$z\Phi(z)$ has degree 2.**

4 Droplet and spherical Schwarz function

- $\partial U: \ z\Phi(z) = \dfrac{|z|^{\frac{2}{r+1}}}{1+|z|^{\frac{2}{r+1}}}$ **is boundary of domain** U**.**
- **If** $\Omega = \{z \mid z^{r+1} \in U\}$ **and** $\boxed{S(z) = z^r \Phi(z^{r+1})}$ **then**

$$\partial\Omega: \quad S(z) = \frac{\bar{z}}{1+|z|^2}$$

  $S(z)$ **is the spherical Schwarz function of** $\partial\Omega$
- **Monotonicity of** Ω **in parameter** t **is crucial to prove that** Ω **is the stereographic projection of the droplet.**

4 Final remark

Thank you for your attention !