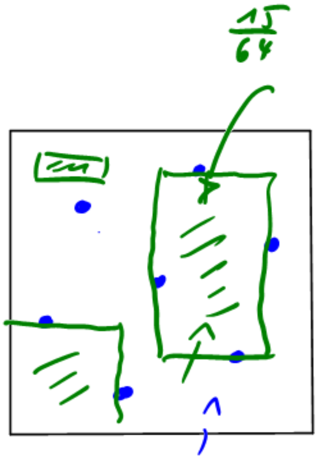
POINT DISTRIBUTION WEBINAR Aug. 25, 2021

DISPERSION - A SURVEY

① Intro:

$\mathcal{P}_n \subseteq [0,1]^d \qquad \# \mathcal{P}_n = n$



$\mathcal{B}$ = set of axis parallel boxes in $[0,1]^d$

$= \{ \prod_{i=1}^{d} [a_i, b_i) : 0 \le a_i < b_i \le 1 \}$

$disp(\mathcal{P}_n) = \sup \{ |B| : B \in \mathcal{B}, B \cap \mathcal{P}_n = \emptyset \}$

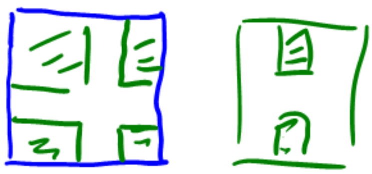
- $disp(n,d) = \inf \{ disp(\mathcal{P}_n) : \# \mathcal{P}_n = n \}$?

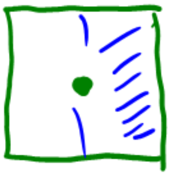
$\to N(\varepsilon, d) = \min \{ n : disp(n,d) \le \varepsilon \}$

- periodic setting $\quad \widetilde{\mathcal{B}}, \widetilde{disp}(\mathcal{P}_n), \widetilde{disp}(n,d)$

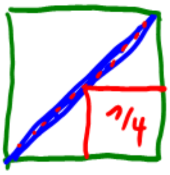
$\widetilde{N}(\varepsilon, d)$

$\widetilde{disp} \ge disp, \quad \widetilde{N} \ge N$





$disp(1,d) = \frac{1}{2} \qquad N(\varepsilon,d) = 1$ if $\varepsilon \ge \frac{1}{2}$



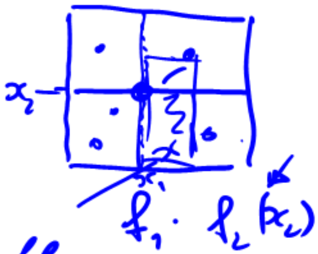
$\frac{1}{4} < \varepsilon < \frac{1}{2}$

APPLIC.: • Combin. Geom. Naamad, Lee, Hsu '84

Dumitrescu, Jiang ≥ 2010.

- Inter. geometric quantity : Hlawka, Rote/Tichy '96
- Optimization : Niederreiter ...
- Approximate rank 1 tensors

$$f = f_1(x_1)\, f_2(x_2) \,..\, f_d(x_d)$$



$f(x) \neq 0$

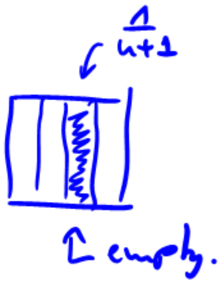
- Marcinkiewicz-type discr. problems. L_p

- lower bound for discrepancy.

Bounds

① $\mathrm{disp}(n,d) \geq \frac{1}{n+1}$

$n\,\mathrm{disp}(n,d) \geq \frac{n}{n+1}$



② $\mathrm{disp}(n,d) \leq \frac{C_d}{n}$

$n\,\mathrm{disp}(n,d) \leq C_d$

Rote / Tichy $\quad 2^d \prod_{i=1}^{d} p_i$

Larcher $\quad 2^{7d+1}$

③ $C_d \geq \frac{1}{8}\log_2 d$

$\rightarrow \geq \frac{\log_2 d}{4}$

AHR

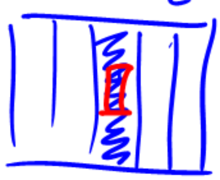
$\leq 2^{7d+2}$

Amplification $\quad \mathrm{disp}(n,d) \geq \frac{\ell+1}{n+\ell+1}\,\mathrm{disp}(\ell,d)$

$\mathrm{disp}(d, 2^d-1) \geq \frac{1}{4}$

$k(\ell-1) < n$

$\frac{\mathrm{disp}(\ell,d)}{k}$



$(n+\ell+1)\,\mathrm{disp}(n,d) \geq (\ell+1)\,\mathrm{disp}(\ell,d)$

$\lim_{n\to\infty} n\,\mathrm{disp}(n,d) = \gamma_d$

$\frac{1}{e} d \leq \gamma_d \leq 8000\, d^2 \log d$

Bukh / Chao

DJ. $\quad$ BH $\quad$ Fibonacci lattices

$d=2$: $\quad \frac{5}{4} \leq \gamma_2 \leq 2 \qquad \tilde{\gamma}_2 \leq 2 \quad \left(\frac{2}{n}\right)$

$\rightarrow 1.50\ldots \leq \gamma_2 \leq 1.89\ldots$

BC $\quad$ KW

M. Ullrich: $\quad \widetilde{\mathrm{disp}}(n,d) \geq \frac{d}{n}$

$\rightarrow \mathrm{disp}(n,d) \geq \frac{\log_2 d}{4}$

Bounds for $N(\varepsilon, d)$

LB | **UB**

$\varepsilon \geq \frac{1}{2}$: LB 1, UB 1

$\frac{1}{4} < \varepsilon < \frac{1}{2}$: UB $1 + \frac{1}{\varepsilon - \frac{1}{4}}$ [S] —3 MacKay

$\frac{11}{\sqrt{\varepsilon - \frac{1}{4}}}$

$\varepsilon < \frac{1}{4}$: LB $\frac{\log_2 d}{\varepsilon}$, UB $C_\varepsilon \log_2 d$

[S] [U-V] [L]

$C_\varepsilon = C \cdot \frac{\ln\left(\frac{e}{\varepsilon}\right)}{\varepsilon^2}$

$C \frac{d^2 \ln d}{\varepsilon}$ BC

$\mathcal{P}_n$ n uniformly distributed points in $[0,1]^d$.

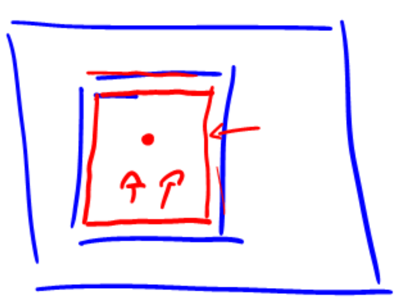
$\frac{8d}{\varepsilon} \log_2\left(\frac{33}{\varepsilon}\right)$ [R]

... [L]

[HKKR]

$\max\left\{\frac{c}{\varepsilon} \ln\left(\frac{1}{\varepsilon}\right), \frac{d}{2\varepsilon}\right\}$

$\frac{d\left(\ln \ln \frac{8}{\varepsilon} + \ln \frac{1}{\varepsilon}\right)}{\varepsilon}$ [LL]



δ-covers

$|B| \geq \varepsilon$

Finite Collection $\mathcal{N}$ $|B| \geq \delta$

$\frac{\ln|\mathcal{N}|}{\varepsilon}$ ← as small as poss.

$N\left(S, -\frac{\gamma}{2} S\right)$

$\varepsilon^{-1/4}$