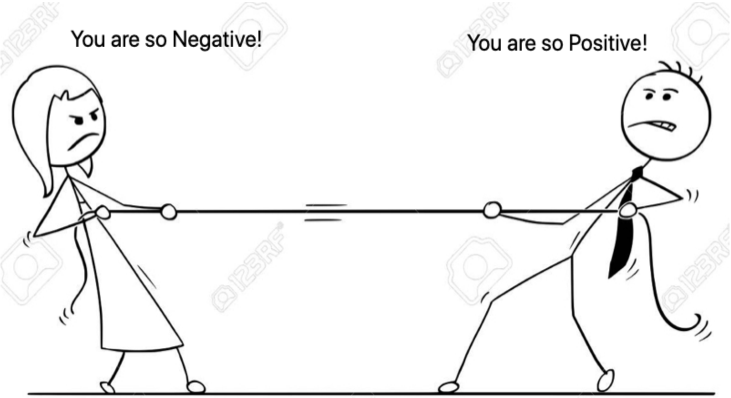
Sign Uncertainty Principle ≈ Tug-of-war



△ Sign uncertainty principle
□ LP bounds for sphere packing
○ Why these problems are so challenging and surprising

We say $f : \mathbb{R}^d \to \mathbb{R}$ is eventually nonnegative (E.NN.) if

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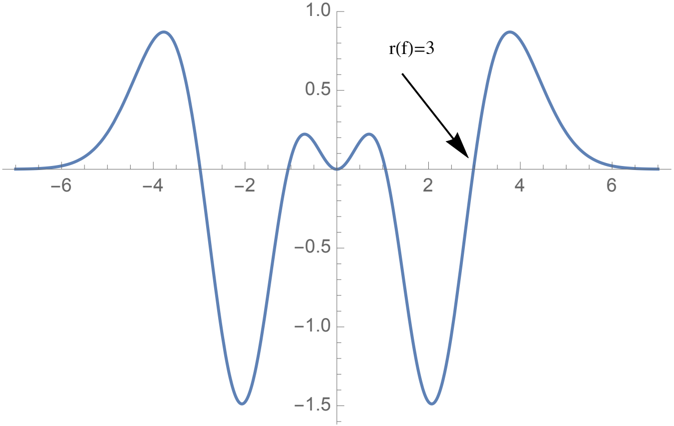

$$\mathcal{A}_+(d) = \begin{cases} f,\ \widehat{f} \in L^1(\mathbb{R}^d) \text{ both even and real-valued} \\ \widehat{f}(0) \le 0,\ f(0) \le 0 \\ f \text{ and } \widehat{f} \text{ E.NN.} \end{cases}$$

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Thm (Bourgain, Clozel, Kahane, 2010)

$$\frac{1}{\sqrt{2\pi e}} \le \frac{\mathbb{A}_+(d)}{\sqrt{d}} \le \frac{1 + o(1)}{\sqrt{2\pi}}.$$

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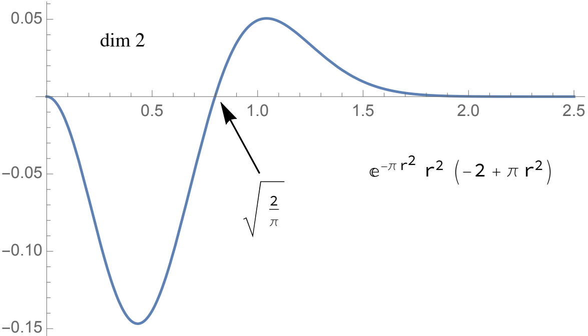

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Thm (Cohn, Gonçalves, 2018)

$$\frac{1}{\sqrt{2\pi e}} \le \frac{\mathbb{A}_-(d)}{\sqrt{d}} \le (1+o(1))\left(\frac{1}{\sqrt{2\pi}} - 0.079\ldots\right).$$

$$\mathbb{A}_-(d) = \inf\{r(f) : f \text{ radial}, \widehat{f} = -f, f(0) = 0, f \text{ E.NN.}\}$$

$\mathbb{A}_\pm(d) \gtrsim \sqrt{d} \rightsquigarrow \pm 1$ **Uncertainty Principles**

Thm (Gonçalves, Oliveira e Silva, Steinerberger, 2016; Cohn, Gonçalves, 2018)

Existence of Optimal: $\exists f \in \mathcal{A}_\pm(d)$ such that

$$r(f) = \mathbb{A}_\pm(d),$$

we can assume f radial, $\widehat{f} = \pm f$ and $f(0) = 0$.

Multiple Roots: $f(|x|)$ has infinitely many double roots for $|x| > r(f)$.

Sphere Packing Problem

What is the most dense arrangement of non-overlapping equal spheres in $\mathbb{R}^d$? $\rightsquigarrow \Delta(d) = \text{largest density}$

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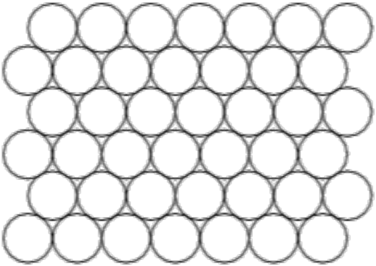
(Thue, 1910)

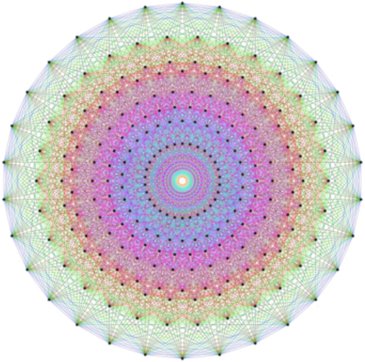


honeycomb ~ 91%

(Hales, 1998)

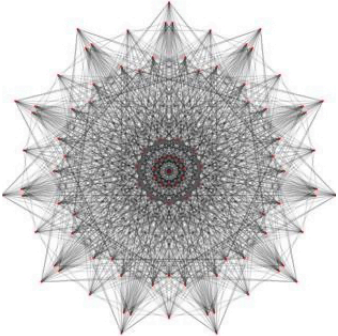
Cannonball Packing ~ 70%

(Viazovska, 2016)



E8 Lattice ~ 25%

(Viazovska, et al 2016)



Leech Lattice Λ_{24} ~ 0.2%

Thm (Cohn, Elkies, 2003) [Linear Programming Bounds]

Let

$$\mathcal{A}_{LP}(d) = \begin{cases} f,\ \widehat{f} \in L^1(\mathbb{R}^d) \text{ radial and real-valued} \\ f(0) = \widehat{f}(0) = 1 \\ f \text{ E.NN. and } \widehat{f} \geq 0. \end{cases}$$

and $\mathbb{A}_{LP}(d) = \inf_{f \in \mathcal{A}_{LP}(d)} r(f)$. Then

$$\Delta(d) \leq vol(\tfrac{1}{2}B^d)\mathbb{A}_{LP}(d)^d.$$

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Viazovska showed $\mathbb{A}_{LP}(8) = \sqrt{2}$ and $\mathbb{A}_{LP}(24) = \sqrt{4}$ with a construction using modular forms.

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- Cohn & G. (2018) showed that
$$\mathbb{A}_+(12) = \sqrt{2}$$
and presented numerical evidence for the conjecture
$$\mathbb{A}_{LP}(d) = \mathbb{A}_-(d) \sim \mathbb{A}_+(d) \sim c\sqrt{d}.$$

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- Recent numerical evidence by Afkhami-Jeddi, Cohn et al in connection with modular bootstraps for CFTs indicates

$$c = \frac{1}{\pi} = .31...$$

d	Best Packing	$\mathbb{A}_{LP}(d)$	$\mathbb{A}_{-}(d)$	$\mathbb{A}_{+}(d)$
1	$\mathbb{Z}$	1	1	??surprise
2	Honeycomb	$? = (4/3)^{\frac{1}{4}}$	$?= (4/3)^{\frac{1}{4}}$	?
8	$E8$	$\sqrt{2}$	$\sqrt{2}$	?
12	?	?	?	$\sqrt{2}$
24	Leech	$\sqrt{4}$	$\sqrt{4}$	?

New Sign Uncertainty Principles (G., Oliveira e Silva, Ramos – arXiv March 2020)

- Spherical Harmonics and Jacobi Polynomials ($\rightsquigarrow$ Spherical Designs and Quadrature).

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- Fourier Series, Bessel-Dini Series.
- Discrete Fourier and Hankel Transf..
- Functions of the Hamming cube (Complexity of Boolean Functions).
- Hankel Transf., Hilbert Transf. and other Smooth Conv. Kernels.

For $f : \mathbb{Z}_{2q+1} \to \mathbb{C}$ we define the DFT

$$\widehat{f}(n) = \ell \sum_{m=-q}^{q} f(n) e^{-2\pi i x \ell m} \Big|_{x=\ell n} \quad (\ell = \frac{1}{\sqrt{2q+1}}).$$

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Thm

Let

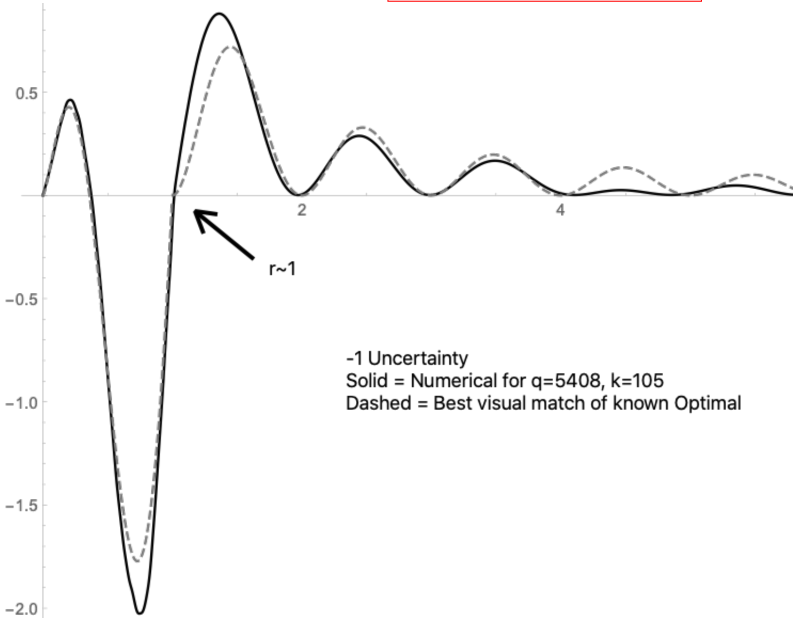
$$\mathcal{A}_{\pm}^{disc}[q] = \begin{cases} f,\ \widehat{f} : \mathbb{Z}_{2q+1} \to \mathbb{R} \text{ both even and real-valued} \\ \widehat{f}(0) \leq 0,\ \pm f(0) \leq 0. \end{cases}$$

Then

$$\mathbb{A}_{\pm}^{disc}[q] := \min\{\sqrt{k(f)\, k(\pm\widehat{f})}\} \gtrsim \sqrt{2q+1},$$

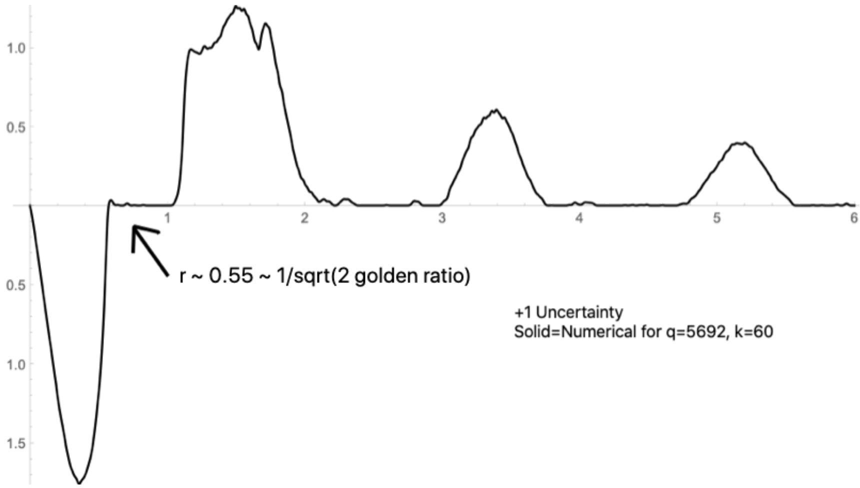
where $k(f) = \min\{k > 0 : f(n) \geq 0 \text{ if } n \geq k\}$.

Numerical evidence that $\frac{\mathbb{A}_{-}^{disc}[q]}{\sqrt{2q+1}} \to \mathbb{A}_{-}(1)$

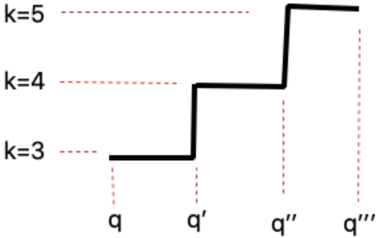


Numerical evidence for

$$\frac{\mathbb{A}_+^{disc}[q]}{\sqrt{2q+1}} \to \frac{1}{\sqrt{2 golden.ratio}} =^{conj.} \mathbb{A}_+(1) \quad (q \to \infty)$$



The function $q \mapsto k = \mathbb{A}_{\pm}^{disc}[q]$ is a stairway: $\boxed{k \mapsto q_{\pm}^{jump}(k)}$



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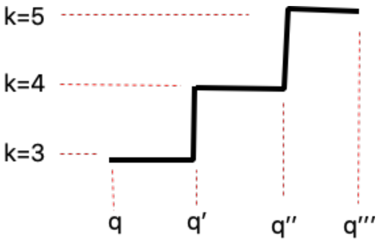

It turns out that (numerically):

$$q^{jump}_{-}(k) = \left\lfloor \frac{k^2 - 2k + 2}{2} \right\rfloor_{k \geq 4} = 5, 8, 13, 18, 25, 32, \ldots$$

$$q^{jump}_{+}(k) \approx \lfloor (k-1)^2 \times \text{golden.ratio} \rfloor_{k \geq 3} = 6, 14, 25, 40, 58, 79, \ldots$$

$$\lim_{q \to \infty} \frac{\mathbb{A}^{disc}_{\mp}[q]}{\sqrt{2q+1}} = \lim_{k \to \infty} \frac{k}{\sqrt{2q^{jump}_{\mp}(k) + 1}} = 1, \quad \frac{1}{\sqrt{2\,\text{golden.ratio}}}$$

In higher dimensions we use a **Disc. Hankel Transf.**

$$H_d^{\text{disc}}(f)(m) = \frac{2}{j_{q+1}} \sum_{n=1}^{q} f(n) \frac{J_{d/2-1}\left(\frac{j_m j_n}{j_{q+1}}\right)}{J_{d/2}(j_n)^2} \quad [j_n = n^{th}\text{-zero of } J_{d/2-1}]$$

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Let

$$\mathcal{A}_{\pm}^{disc}[d,q] = \begin{cases} f,\ H_d f : \{1,..,q\} \to \mathbb{R} \\ H_d f(0) \le 0,\ \pm f(0) \le 0. \end{cases}$$

and

$$\mathbb{A}_{\pm}^{disc}[d,q] := \min\{\sqrt{k(f)\,k(\pm H_d f)}\}.$$

Then $j_{\mathbb{A}_{\pm}^{disc}[d,q]} \gtrsim \sqrt{2\pi j_{q+1}}$

$$\frac{j_{\mathbb{A}_{\pm}^{disc}[d,q]}}{\sqrt{2\pi j_{q+1}}} \to \mathbb{A}_{\pm}(d) \text{ numerically } (q \to \infty)$$

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$$q_{-}^{jump}(\mathbf{2}, k) \approx \left\lfloor \frac{\sqrt{3}(k^2-2k+2)}{4} \right\rfloor_{k\geq 4} = 4, 7, 11, 16, 21, 28, 35, 43, 52, 62, \ldots$$

$$q_{-}^{jump}(\mathbf{8}, k) \approx \left\lfloor \frac{k^2}{4} \right\rfloor_{k\geq 4} = 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, \ldots,$$

$$q_{-}^{jump}(\mathbf{24}, k) \approx \left\lfloor \frac{k^2+6k-8}{8} \right\rfloor_{k\geq 4} = 4, 5, 8, 10, 13, 15, 19, 22, 26, 29, \ldots,$$

$$q_{+}^{jump}(\mathbf{12}, k) \approx \left\lfloor \frac{k^2+2k-1}{4} \right\rfloor_{k\geq 3} = 3, 5, 8, 11, 15, 19, 24, 29, 35, 41, \ldots,$$

numerically

$$\frac{j_{\mathbb{A}_-^{disc}[d,q]}}{\sqrt{2\pi j_{q+1}}} \to 1,\ (\tfrac{4}{3})^{\frac{1}{4}},\ \sqrt{2},\ \sqrt{4} \quad (d = 1, 2, 8, 24)$$

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There are Thm's to be proven here!