Configurations, Automorphisms and Cohomology

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Configurations

Definition

A configuration is a complex matrix $\Phi \in \mathbb{C}^{d \times n}$. We view it as an ordered list of vectors $\{\Phi_i\}_{i=0}^{n-1}$ in $\mathbb{C}^d$.
$d =$ The dimension,
$n =$ The cardinality.

Some Definitions:

- Φ is a Tight-Frame (TF) if for every vector v,

$$v = \sum_{i=0}^{n-1} \langle v, \Phi_i \rangle \Phi_i.$$

  Equivalently, Φ can be completed to a $n \times n$ orthogonal matrix.
- A unit norm TF Φ is m-angular if $\{|\langle \Phi_i, \Phi_j \rangle|\}_{i<j}$ has cardinality m.

m-angular tight-frames

- If $m = 1$, then Φ is called an Equiangular-Tight-Frame (ETF).
- m-angular TF arise often as minimizers of potential functions. E.g. the Frame-Potential given by

$$FP_p(\Psi) = \sum_{i,j} |\langle \Psi_i, \Psi_j \rangle|^p, \text{ s.t. } \forall i \ ||\Psi_i|| = 1.$$

- In the special case $n = d^2$, the ETF Φ is called a SIC-POVM. In is conjectured (Zauner) to exist for all $d \geq 1$.
- A set of Mutually Unbiased Bases (MUB) is $m = 2$-angular: We have $n = rd$ and

$$|\langle \Phi_i, \Phi_j \rangle|^2 = \begin{cases} 1 & i = j \\ 0 & sd \leq i < j < (s+1)d \\ 1/d & \text{otherwise} \end{cases} .$$

Algebraic Configurations

Fix an algebraic closure $\overline{\mathbb{Q}}/\mathbb{Q}$.

Definition

(a) A configurations Φ is Algebraic if all $\Phi_{i,j} \in \overline{\mathbb{Q}}$.

(b) Φ is Potentially Algebraic (PA) if there are phases $e^{t_j\sqrt{-1}}$, $0 \leq j < n$ and a unitary $U \in U(d)$ such that

$$e^{t_j\sqrt{-1}} U\Phi_j \in \overline{\mathbb{Q}}^d.$$

- Often, minimaizers of potential functions solve algebraic equations and thus are algebraic.
- The known examples of Zauner SIC-POVMs and maximal MUBs are potentially algebraic.
- The entries of known algebraic MUBs are roots of unity.

Algebraic Configurations

- The entries of known algebraic WH SIC-POVMs are is an abelian extension field (Appleby, 2012)

$$E \;/\; \mathbb{Q}(e^{2\pi\sqrt{-1}/d}, \sqrt{(d-1)(d+3)}).$$

- Observation: Φ is potentially algebraic $\iff$ there is a phase diagonal matrix D s.t.

$$D^*(\Phi^*\Phi)D \;\text{ is algebraic.}$$

Automorphism Groups

The following actions on Φ preserve the multiset $\{|\langle \Phi_i, \Phi_j \rangle|\}$:

(i) Phasing: for $\alpha = (\alpha_i) \in \mathbb{R}^n$, Let $(\alpha * \Phi)_i := e^{\alpha_i \sqrt{-1}} \Phi_i$.

(ii) Permutations: For $\pi \in S_d$, Let $(\pi * \Phi)_i := \Phi_{\pi^{-1}(i)}$.

(iii) Rotation: For $U \in U(d)$, Let $U * \Phi := U\Phi$.

(iv) Complex Conjugation: Let $(conj * \Phi) := \overline{\Phi}$.

The combination of all such actions on the space of configurations **Conf**(d, n) form a group, denoted by $\Sigma(d, n)$.

Definition

The Automorphism Group of Φ is the group

$$\mathrm{Aut}(\Phi) := \{g \in \Sigma(d, n) \mid g * \Phi = \Phi\}.$$

Extended automorphism group

Let Φ be algebraic and let $Gal(\overline{\mathbb{Q}}/\mathbb{Q})^0 \subset Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ be the centralizer of complex conjugation. We can add

(v) Galois Conjugation: For $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})^0$, let $(\sigma * \Phi) := \Phi^\sigma$.

Let $\Sigma^*(n, d) \supset \Sigma(n, d)$ be the group of actions on algebraic configurations generated by (i)-(v).

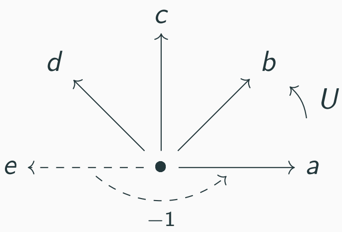
Definition

For an algebraic Φ, the Extended Automorphism Group is the group

$$\mathrm{Aut}^*(\Phi) := \{g \in \Sigma^*(d, n) \mid g * \Phi = \Phi\}.$$

An Example

Let $\Phi = [a, b, c, d] \in \mathbf{Conf}(2, 4)$ (defined over $\mathbb{Q}(\sqrt{2})$) as in the picture.



- Here is an automorphism and an extended automorphism. Apply:

$$[a, b, c, d] \xrightarrow{U} [b, c, d, -a] \xrightarrow{phase} [b, c, d, a] \xrightarrow{perm} [a, b, c, d],$$

$$[a, b, c, d] \xrightarrow{Galois} [a, -b, c, -d] \xrightarrow{phase} [a, b, c, d].$$

Automorphism group on the Grammian Matrix

Assume $rank(\Phi) = d$. The Grammian $G(\Phi) := \Phi^*\Phi$ determines Φ up to rotation. On

$$\mathcal{G}(n,d) := \{\Phi^*\Phi \mid \Phi \in \mathbf{Conf}(n,d)\}$$

the actions (i)-(v) reduce to the form

(i)+(ii) Monomial Matrices: $M * G := MGM^*$.

(iv)+(v) Galois Conjugation $g * G := G^g$.

We denote the groups by $\mathrm{Aut}(G)$ and $\mathrm{Aut}^*(G)$.

- We have $\mathrm{Aut}(G(\Phi)) = \mathrm{Aut}(\Phi)$ and $\mathrm{Aut}^*(G(\Phi)) = \mathrm{Aut}^*(\Phi)$.
- If G is not a nontrivial diagonal sum of blocks, then $\mathrm{Aut}(G) \subset \mathrm{Aut}^*(G)$ are finite.

Example - The DetFourier Matrix

Consider the $n^2 \times n^2$ matrix indexed by $\mathbb{Z}/n \oplus \mathbb{Z}/n$ in both axes. Let $\omega = \exp(2\pi\sqrt{-1}/n)$, and

$$DF(u,v) = \frac{\omega^{\det(u,v)}}{n}.$$

(i) $\forall w, \quad DF(u+w, v+w) = DF(u,w)DF(u,v)\overline{DF(v,w)}.$

(ii) $\forall A \in GL_2(\mathbb{Z}/n), \quad DF(Au, Av) = DF(u,v)^{\det(A)}.$

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$$AGL_2(\mathbb{Z}/n) = GL_2(\mathbb{Z}/n) \ltimes (\mathbb{Z}/n)^2 \longrightarrow \mathrm{Aut}^*(DF).$$

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The same automorphisms are shared by

$$(I_{n^2} - DF)/2 = \Phi^*\Phi, \quad \Phi \in \mathrm{Conf}((n^2-n)/2, n^2).$$

Example - Gabor Frames

The modulation and translation matrices are $M = \mathrm{diag}(\dots, \tau^i, \dots)$ and $T \in U(n)$ given by

$$T_{i,j} = \begin{cases} 1 & i - j \equiv 1 \mod n \\ 0 & \text{otherwise} \end{cases}.$$

We have

$$MT = \omega TM.$$

Definition

The Weil-Heisenberg group is

$$WH(n) := \{\tau^a M^b T^c \mid a, b, c \in \mathbb{Z}/n\}.$$

- Let $g \in \mathbb{C}^n$, $||g|| = 1$. The Gabor frame is

$$\mathcal{G}(g) = (M^b T^c g)_{b,c \in \mathbb{Z}/n}.$$

Gabor Frames

g is called the fiducial vector of $\mathcal{G}(g)$. We have

$$WH(n) \mapsto \mathrm{Aut}(\mathcal{G}(g)),$$

given by $g \mapsto (x \mapsto gx)$.

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- Zauner SIC-POVMs are of type $\mathcal{G}(g)$ for some fiducial g.
- The normalizer of $WH(n)$ in $U(n)$ is called the Extended Clifford Group, $EC(n)$.
- Known Zauner SIC-POVMs ($d \neq 2, 3, 8$) admit an order 3 automorphism in $EC(n)$ (augmented with phases and permutations).
- Known Zauner SIC-POVMs are algebraic. They have extended automorphisms (coming from Multiplets)

Abstract Automorphisms on Grammians

- Let X be a finite set. Let $Mon(X)$ be the group of phase monomial matrices on X. e.g. matrices $M_{j,\pi(j)} = \alpha_j$, $|\alpha_j| = 1$ and $\pi \in S_X$ a permutation.
- Let G be a finite group.

Definition

An abstract automorphism group on X is a map $\varphi : G \to Mon(X)$ which is a homomorphism up to a phase:

$$\varphi(gg') = \alpha\varphi(g)\varphi(g'), \quad |\alpha| = 1.$$

Observation

The set of all matrices in $M \in \mathbb{C}^{X\times X}$ satisfying

$$\varphi(g)M\varphi(g)^* = M, \quad \forall g \in G$$

is a matrix algebra. We denote it by $\mathcal{A}(\varphi)$.

Abstract Automorphisms on Grammians

Let $\varphi : G \to Mon(X)$ be an abstract automorphism group, and let $M \in \mathcal{A}(\varphi)$ be a matrix.

- If M is self-adjoint positive semidefinite, then $M = \Phi^*\Phi$ for a configuration Φ. We have a representation

$$G \to \mathrm{Aut}(\Phi).$$

- If in addition M is an idempotent, then Φ is a tight frame.

Extended Abstract Automorphisms on Grammians

Let $K \subset \mathbb{C}$ be a number-field, Galois over $\mathbb{Q}$, and let $\mu \subset K^\times$ be a subgroup of unit norm elements. Let

$Mon(X, \mu) \subset Mon(X)$ the subgroup of μ-valued mon. matrices.

Definition

An Extended Abstract Automorphism Group on X is a pair (γ, φ) such that

1. $\gamma : G \to Gal(K/\mathbb{Q})$ is a homomorphism preserving μ.
2. $\varphi : G \to Mon(X, \mu)$ is a map satisfying

$$\varphi(gg') \doteq \varphi(g)\varphi(g')^{\gamma(g)}.$$

($\doteq$ is equality up to phases in μ.)

Algebraic Matrix Algebras

- Given an extended automorphism Group (φ, γ), the set

$$\mathcal{A}(\varphi,\gamma) := \{M \in K^{X\times X} \mid \varphi(g)\, M^{\gamma(g)}\, \varphi(g)^* = M,\ \forall g \in G\}$$

  is a matrix algebra over $\mathbb{Q}$.
- Again, self-adjoint idempotents correspond to algebraic TF Φ, admitting a homomorphism $G \to \mathrm{Aut}^*(\Phi)$.

Examples

Let B_3= the symmetry group of the 3D cube.
X= the set of all faces.
Define $\varphi : G \to Mon(X)$ = the permutation action on X.

$$\mathcal{A}(\varphi) = \left\{ \begin{pmatrix} x_0 & x_2 & x_2 & x_2 & x_2 & x_1 \\ x_2 & x_0 & x_2 & x_2 & x_1 & x_2 \\ x_2 & x_2 & x_0 & x_1 & x_2 & x_2 \\ x_2 & x_2 & x_1 & x_0 & x_2 & x_2 \\ x_2 & x_1 & x_2 & x_2 & x_0 & x_2 \\ x_1 & x_2 & x_2 & x_2 & x_2 & x_0 \end{pmatrix} \right\}.$$

$x_0 = x_3 = 1/3, x_2 = -1/6$ gives a self-adjoint idempotent of rank 2. The configuration is the perfect hexagon (up to phases).

Constructing Automorphisms from Group Theory

We need:

- A finite group G and a left action $G \curvearrowright X$.
- A number field $K/\mathbb{Q}$ and a homomorphism $\gamma : G \to Gal(K/\mathbb{Q})$.
- A subgroup of phases $\mu \subset K^\times$, stable under $\gamma(G)$.

$\implies$ we can define an action of G on $K^{X \times X}$:

$$(gM)_{x,y} := (M_{g^{-1}x, g^{-1}y})^{\gamma(g)}.$$

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$\implies$ we can define an action of G on $K^{X \times X}$:

$$(gM)_{x,y} := (M_{g^{-1}x, g^{-1}y})^{\gamma(g)}.$$

- Let $\pi(g)$ be the permutation of g on X. Then

$$\mathcal{A}(\pi, \gamma) = \{M \mid gM = M\}.$$

To give X, is the same as to give a list of subgroups $\{H_i \subset G\}$, and then

$$X \cong \bigsqcup_i G/H_i.$$

So this is completely group-theoretic.

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- We want even more: To consider monomial actions.
- This means that π should be replaced by $\varphi : G \to Mon(X, \mu)$, satisfying the conditions explained above.
- We want φ to lift π, i.e.

$$|\varphi(g)| = \pi(g), \quad \text{(entrywise modulus)}.$$

- Here is where Cohomology comes in ...

We need

$$\varphi(gg') \doteq \varphi(g)\varphi(g')^{\gamma(g)} \tag{1}$$

$$|\varphi(g)| = \pi(g) \tag{2}$$

Let's try an easier condition first:

$$\varphi(gg') = \varphi(g)\varphi(g')^{\gamma(g)} \tag{3}$$

Write $\varphi(g) = \delta(g)\pi(g)$, $\delta(g)$ diagonal. Then

$$\varphi(gg') = \delta(gg')\pi(gg') = \delta(gg')\pi(g)\pi(g')^{\gamma(g)}$$

$$\begin{aligned}\varphi(g)\varphi(g')^{\gamma(g)} = (\delta(g)\pi(g))(\delta(g')\pi(g'))^{\gamma(g)} = \\ = \delta(g)\pi(g)\delta(g')^{\gamma(g)}\pi(g')^{\gamma(g)} \\ = \delta(g)\pi(g)\delta(g')^{\gamma(g)}\pi(g)^{-1}\pi(g)\pi(g')^{\gamma(g)}.\end{aligned}$$

Equating,

$$\delta(gg') = \delta(g)\pi(g)\delta(g')^{\gamma(g)}\pi(g)^{-1}.$$

We see that δ is a 1-cocycle. More precisely, let

$$D_X(\mu) = \{X \times X \text{ Diagonal matrices over } \mu\}.$$

Let $G \curvearrowright D(\mu)$ by its action on $K^{X \times X}$. The δ defines a cohomology class in

$$[\delta] \in H^1(G, D_X(\mu)).$$

So we want to compute this cohomology.

Theorem (Eckmann-Shapiro Lemma)

If $X \cong G/H$, then

$$H^1(G, D_X(\mu)) \cong H^1(H, \mu).$$

More generally, if $X \cong \bigsqcup_i G/H_i$, then

$$H^1(G, D_X(\mu)) \cong \bigoplus_i H^1(H_i, \mu).$$

Cohomology Basics

Let G be a group, acting on an Abelian group A. For any function

$$f : G^n \to A,$$

let $d_i f : G^{n+1} \to A$ given by

$$d_i f(g_0, g_1, \ldots, g_n) = \begin{cases} f(g_0, \ldots, g_{n-1}) & i = n \\ f(g_0, \ldots, g_i \cdot g_{i+1}, \ldots, g_n) & 0 < i < n \;, \\ g_0 f(g_1, \ldots, g_n) & i = 0. \end{cases}$$

and let

$$df = \sum_{i=0}^{n} (-1)^i d_i f, \quad d \circ d = 0.$$

Definition

1. f is a cocycle if $df = 0$,
2. f is a coboundary if $f = dh$.

Coboundaries $\subseteq$ Cocycles.

Cohomology Basics

- The nth-Cohomology Group is

$$H^n(G, A) = \frac{n - \text{cocycles}}{n - \text{coboundaries}}.$$

- If $F \leq G$ is is subgroup of G, then there is a natural homomorphism

$$res : H^n(G, A) \to H^n(F, A),$$

given by restriction of cocycles.

0,1,2-Cohomology

- $H^0(G, A) = A^G :=$ The group of G-invariant elements.
- 1-cocycles are functions $z : G \to A$ such that

$$z(gg') = z(g) + gz(g').$$

- 1-coboundaries are functions $z : G \to A$ such that

$$z(g) = ga - a, \text{ for a given } a \in A.$$

- If G acts trivially on A, then

$$H^1(G, A) \cong Hom(G, A).$$

0,1,2-Cohomology

- 2-cocycles are functions $z : G^2 \to A$ such that

$$g_0 z(g_1, g_2) + z(g_0, g_1 g_2) = z(g_0 g_1, g_2) + z(g_0, g_1).$$

- 2-coboundaries are functions $z : G^2 \to A$ such that

$$z(g_1, g_2) = u(g_1 g_2) - u(g_1) - g_1 u(g_2).$$

  for some function $u : G \to A$.

H^1 - Development

Recall: $G \curvearrowright X$, $\pi(g)$ is the permutation matrix on X, we write

$$\varphi(g) = \delta(g)\pi(g), \quad \delta(g) \in D_X(\mu) \text{ i.e. diagonal over } \mu,$$

$$\varphi(gg') = \varphi(g)\varphi(g')^{\gamma(g)}. \text{ (φ is called H^1-Develpoed action)}$$

Then δ is a 1-cocycle w.r.t. $G \curvearrowright D_X(\mu)$:

$$\delta(gg') = \delta(g)\pi(g)\delta(g')^{\gamma(g)}\pi(g)^{-1}.$$

Observation

Let $\varphi'(g) = \delta'(g)\pi(g)$ and $\varphi(g) = \delta(g)\pi(g)$. Then $[\delta'] = [\delta] \in H^1(G, D_X(\mu))$,

$$\iff \exists D \in D_X(\mu), \ \varphi' = D\varphi D^{-1}.$$

H^2-Developement

Suppose that we only want

$$\varphi(gg') \doteq \varphi(g)\varphi(g')^{\gamma(g)}.$$

Write

$$\varphi(gg') = \alpha(g,g')\varphi(g)\varphi(g')^{\gamma(g)}, \quad \alpha(g,g') \in \mu.$$

- Then $\alpha(g,g')$ is a 2-cocycle over μ.
- If φ and φ' admit the same class $[\alpha] \in H^2(G,\mu)$, then

$$\varphi'(g) = \delta_0(g)\varphi(g), \quad [\delta_0] \in H^1(G, D_X(\mu)).$$

- Monomial actions φ are controlled by $H^1(G, D_X(\mu))$ and $H^2(G,\mu)$.

A Program for computing Symmetric Algebraic Configurations

(i) Choose a group G and an action $G \curvearrowright X$.

(ii) Choose a Number Field K, a Galois action $\gamma : G \to Gal(K/\mathbb{Q})$ and a G-stable phase group $\mu \in K^\times$.

(iii) Compute a monomial action $\varphi : G \to Mon(X, \mu)$ using cohomology.

(iv) Construct a basis to the algebra $\mathcal{A}(\varphi, \gamma)$.

(v) Study it's structure and compute self-adjoint idempotents. Generate configurations.

- The number of angles $+1 \leq$ the number of G-orbits in $X \times X$.

Constructing a basis to $\mathcal{A}(\varphi, \gamma)$.

$\mathcal{A}(\varphi, \gamma)$ is an algebra over K^G, the subfield of G-invariants.

(a) Compute the orbit decomposition $X \times X = \bigsqcup_i O_i$.

- Pick a point $(x_i, y_i) \in O_i$ for each i.
- For each i, pick a suitable value $\xi_i \in K^\times$.
- For each i there is at most one matrix $B_i \in K^{X \times X}$, supported in O_i, having

$$(B_i)_{x_i, y_i} = \xi_i.$$

- The collection $\{B_i\}$ spans over K^G the algebra $\mathcal{A}(\varphi, \gamma)$.

Orientability of Orbits

Definition

An orbit $O \subset X \times X$ is orientable if for each point $(x, y) \in O$ there exist a $\xi \in K^\times$ such that

$$\varphi(h)_{x,x}\xi^{\gamma(h)}\varphi(h)^*_{y,y} = \xi, \quad (4)$$

for all $h \in G$ s.t. $hx = x, hy = y$.

Meaning: The monomial action φ does not destroy the value ξ of a putative matrix $A \in \mathcal{A}(\varphi, \gamma)$.

- Basis matrices B_i exist only for orientable orbits B_i.
- Elements of $\mathcal{A}(\varphi, \gamma)$ must vanish at non-orientable orbits.
- The value ξ must be chosen carefully in order to comply with condition (4).
- The appropriate element ξ can be found via cohomology.

The Spectral Sequence

Let G, X, γ, μ, K be given.

Definition

Two matrices $A, B \in K^{X \times X}$ are phase-equivalnt, written

$$A \sim_P B \iff A = DBD^{-1}, \quad \text{for } D \in D_X(\mu).$$

Definition

A matrix $A \in K^{X \times X}$ is Cohomology-Developed (CDM) w.r.t. (G, X, γ, μ, K), if

$$\forall g \in G, \quad gA \sim_P A.$$

The Spectral Sequence

$$H^0(G,(K^\times)^{X\times X}/\sim_P) \;=\; \{\text{CDMS with nonzero entries}\}/\sim_P .$$

- Wish to compute $H^0(G,(K^\times)^{X\times X}/\sim_P)$.
- This can be done in terms of a spectral sequence.

Suppose that

$$X \cong \bigsqcup_i G/H, \tag{5}$$

$$X\times X \cong \bigsqcup_i G/F_i, \tag{6}$$

where $F_i \subset H$.

The Spectral Sequence

Theorem

There exists a first quadrant cohomological spectral sequence $E_1^{i,j} \implies H^{i+j}(G, M_{X\times X}/\sim_P)$ *whose* E_1*-page is:*

$$H^2(G,\mu) \xrightarrow{res_H^G} H^2(H,\mu)$$

$$H^1(G,\mu) \longrightarrow H^1(H,\mu) \xrightarrow{\oplus res_{F_i}^H} \bigoplus_i H^1(F_i, K^\times) \cdot$$

$$H^0(H,\mu) \longrightarrow \bigoplus_i H^0(F_i, K^\times)$$

Orientability Conditions

$$
\begin{array}{ccccc}
H^2(G,\mu) & \xrightarrow{res_H^G} & H^2(H,\mu) & & \\
H^1(G,\mu) & \longrightarrow & H^1(H,\mu) & \xrightarrow{\oplus res_{F_i}^H} & \bigoplus_i H^1(F_i,K^\times) \\
 & & H^0(H,\mu) & \longrightarrow & \bigoplus_i H^0(F_i,K^\times)
\end{array}
\cdot
$$

Orientability Conditions

$$
\begin{array}{ccccc}
H^2(G,\mu) & \xrightarrow{res^G_H} & H^2(H,\mu) & & \\
H^1(G,\mu) & \longrightarrow & H^1(H,\mu) & \xrightarrow{\oplus res^H_{F_i}} & \bigoplus_i H^1(F_i,K^\times) \\
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\end{array}
.
$$

Orientability Conditions

H^2-data

$$H^2(G,\mu) \xrightarrow{res^G_H} H^2(H,\mu)$$

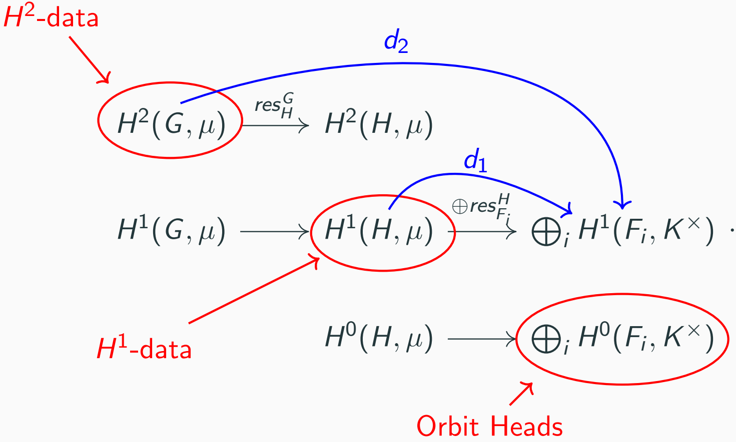
$$H^1(G,\mu) \longrightarrow H^1(H,\mu) \xrightarrow{\oplus res^H_{F_i}} \bigoplus_i H^1(F_i,K^\times) \; .$$

$$H^0(H,\mu) \longrightarrow \bigoplus_i H^0(F_i,K^\times)$$

H^1-data

Orbit Heads

Orientability Conditions



d_1, d_2=Orientability Obstructions

A formula for generating CDMs

We assume the $X \cong G/H$ is G-transitive. We need the following ingredients:

H^2-Data: A class in $[\alpha] \in H^2(G, \mu)$ such that

$$res_H^G([\alpha]) = [0] \in H^2(H, \mu).$$

- A trivialization $\lambda : H \to \mu$ such that $\alpha|_{H \times H} = d\lambda$.

H^1-Data: A class $[\varepsilon] \in H^1(H, \mu)$.

(i) Choose Coset Representatives $\{g_i\}$ for G/H.

(ii) Define a map $for : G \to H$ such that

$$g = for(g) g_i^{-1}, \quad \text{for a representative } g_i.$$

A formula for generating CDMs

Let

$$\delta_{g,i} := \left(\frac{\varepsilon(h)\lambda(h)\alpha(g_i^{-1}, g)}{\alpha(h, 1)} \right)^{\gamma(g_i)},$$

where

$$h = for(g_i^{-1} g).$$

- Finally, let

$$\delta(g) = diag(\delta_{g,i})_i.$$

Then

$$\varphi(g) = \delta(g)\pi(g)$$

gives the desired monomial action.

Cocyclic Matrices

In the special case $X = G$ (action by left multiplication), the resulting CDMs take the simple form

$$M = M(f) = \left(\frac{\alpha(x^{-1}y, y^{-1})^{\gamma(x)}}{\alpha(1, y^{-1})^{\gamma(y)}} \cdot f(x^{-1}y)^{\gamma(x)} \right)_{x,y \in G},$$

where $f : G \to K$ is an arbitrary function. We have

$$\mathcal{A}(\varphi, \gamma) = \{ M(f) \mid f : G \to K \},$$

an algebra over K^G.

- $M(f)$ is called a Cocyclic matrix.
- We will address $\mathcal{A}(\varphi, \gamma)$ as the Cocyclic Algebra.

Cocyclic Algebras

Cocyclic Algebras is already a rich source of examples of configuration algebras. Let $\omega = \exp(2\sqrt{-1}\pi/n)$.

- Gabor frames are a special case. Take $G = \mathbb{Z}/n \times \mathbb{Z}/n$, the 2-cocycle $\alpha((a,b),(c,d)) = \omega^{bc}$, and $\gamma = id$. We have

  $$\mathcal{A}(\varphi, \gamma) \cong M_n(\mathbb{Q}(\omega)).$$

  Minimal idempotents correspond to Gabor frames.
- Taking $\alpha = 1$ instead, gives a different algebra:

  $$\mathcal{A}(\varphi', \gamma) \cong \mathbb{Q}(\omega)^{n^2}.$$

Examples (Twisted Gabor Frames)

Let

$$\alpha((a,b),(c,d)) := \omega^{bc+\chi(a,c)},$$

where

$$n \cdot \chi(a,c) := \widehat{a+c} - \widehat{a} - \widehat{c},$$

$$0 \le \widehat{(x \mod n)} < n, \quad \widehat{(x \mod n)} \equiv x \mod n.$$

- This leads to configurations $\Phi \in \mathbf{Conf}(d, d^2)$.
- They are not phase equivalent to Gabor frames. They are generated by certain two unitary matrices $\tilde{M}, \tilde{T}$ with

$$\tilde{M}^9 = \tilde{T}^3 = I; \quad \tilde{M}\tilde{T} = \omega \tilde{T}\tilde{M}.$$

- One cannot replace $\tilde{M} = \omega^{1/3} M$, $\tilde{T} = T$.

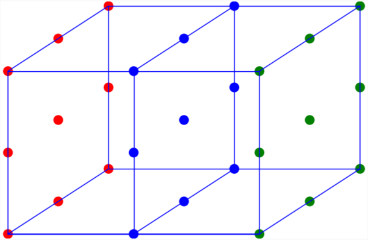
Examples (Gabor Cubes (Conjectural))

Let $G = \mathbb{Z}/n \times \mathbb{Z}/n \times \mathbb{Z}/n$. Let

$$\alpha((v_0, v_1, v_2), (u_0, u_1, u_2)) = \omega^{v_0 u_1 + v_1 u_2 - v_2 u_0}.$$

- Conjecture:
$$\mathcal{A}(\varphi, \gamma) \cong M_n(\mathbb{Q}(\omega))^n.$$
- The minimal idempotents are of rank n. They correspond to $\Phi \in \mathbf{Conf}(n, n^3)$.
- They form a Gabor Cube (see figure).

Gabor Cube (Conjectural)



- Every layer is a $n \times n^2$ Gabor frame.
- Moreover, this is true for many 2-dim affine subspaces of $(\mathbb{Z}/n)^3$.
- The system is generated by 3 unitary operators M_1, M_2, M_3 with $M_i^n = I$ and

$$\forall i, \quad M_i M_{i+1} = \omega M_{i+1} M_i.$$

Example: A_5

Let $G = A_5$, the alternating group, and let

$$G \curvearrowright X := \{\{i,j\} \mid 1 \leq i < j \leq 5\}.$$

- We first choose $\alpha = \varepsilon = 0$. We get a permutation action on $\mathbb{C}^{X \times X}$.

$$\mathcal{A}(\varphi, \gamma) \cong \mathbb{Q}^3.$$

- There are 3 minimal idempotents, E_0, E_1, E_2 of ranks $1, 4, 5$.

Example: A_5

The configuration in **Conf**$(4, 10)$ has Gram matrix

$$G(\Phi) = \begin{pmatrix} 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 1 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 1 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & 1 & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 1 & \frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 1 & -\frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & 1 \end{pmatrix}.$$

Example: A_5

On the other hand, $X \cong G/H$ with $H \cong S_3$, and choose

$$\varepsilon : H \to \{\pm 1\}, \quad \varepsilon = sign.$$

- $$\mathcal{A}(\varphi, \gamma) \cong \mathbb{Q} \times \mathbb{Q}(\sqrt{5}).$$
- There are 3 idempotents, E_0, E_1, E_2 of ranks $4, 3, 3$. The latter two are Galois conjugates.

Example: A_5

The Gram matrix of the resulting $\Phi \in \mathbf{Conf}(4,10)$ is

$$G(\Phi) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix}.$$

THANK YOU