# MAPPING TO THE SPACE OF SPHERICAL HARMONICS

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# PLAN

1. Mapping to spherical harmonics
2. Packing problems
3. Energy bounds
4. Constructing new configurations
5. Kissing number problem in dimension 3

# SPHERICAL HARMONICS

A polynomial $P : \mathbb{R}^d \to \mathbb{C}$ is harmonic if $\Delta P = 0$, where $\Delta$ is a Laplacian.

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The restriction of a harmonic homogeneous polynomial P of degree k to $\mathbb{S}^{d-1}$ is called a spherical harmonic of degree k.

The vector space of spherical harmonics of degree k, $\mathrm{Harm}_k(\mathbb{S}^{d-1})$, has dimension

$$h_k = \binom{d+k-2}{k} + \binom{d+k-3}{k-1},$$

and a natural scalar product

$$\langle P, Q \rangle = \int_{\mathbb{S}^{d-1}} P(x)\overline{Q(x)} d\mu(x),$$

where $\mu$ is the normalized Lebesgue measure of the sphere.

# DEFINING A MAPPING TO SPHERICAL HARMONICS

Let $\{e_1^{(k)}, \ldots, e_{h_k}^{(k)}\}$ be an orthonormal basis of $\mathrm{Harm}_k(\mathbb{S}^{d-1})$. Define a map $\phi_k : \mathbb{S}^{d-1} \to \mathbb{C}^{h_k}$ by

$$\phi_k(x) = \frac{1}{\sqrt{h_k}}(e_1^{(k)}(x), \ldots, e_{h_k}^{(k)}(x)).$$

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It appears $\langle \phi_k(x), \phi_k(y) \rangle = \frac{1}{h_k} \sum_{i=1}^{h_k} e_i^{(k)}(x)\overline{e_i^{(k)}(y)}$ depends only on $\langle x, y \rangle$ and is the same for any choice of the orthonormal basis $\{e_i^{(k)}\}$.

All scalar products $\langle \phi_k(x), \phi_k(y) \rangle$ are real and $\langle \phi_k(x), \phi_k(x) \rangle = 1$ for all $x \in \mathbb{S}^{d-1}$. Therefore, $\phi_k$ maps $\mathbb{S}^{d-1}$ to $\mathbb{S}^{h_k - 1}$.

$$\langle \phi_k(x), \phi_k(y) \rangle = G_k(\langle x, y \rangle),$$

where $G_k$ are Gegenbauer polynomials, zonal spherical functions associated with $\mathrm{Harm}_k(\mathbb{S}^{d-1})$.

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$G_1(t) = t$

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$\phi_k$ is similar to the tensor power map $x \to x^{\otimes k}$.

$\phi_k$ is a map from $\mathbb{S}^{d-1}$ to $\mathbb{S}^{h_k-1}$ such that for any x and y,

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For $P(t) = \alpha_1 G_{i_1}(t) + \ldots + \alpha_l G_{i_l}(t)$, where $\alpha_i > 0$ and $\sum \alpha_i = 1$, define

$$\phi_P(x) = \sqrt{\alpha_1}\phi_{i_1}(x) \oplus \ldots \oplus \sqrt{\alpha_l}\phi_{i_l}(x).$$

$\phi_P$ maps $\mathbb{S}^{d-1}$ to $\mathbb{S}^{H-1}$, where $H = h_{i_1} + \ldots + h_{i_l}$, and for any x and y,

$$\langle \phi_P(x), \phi_P(y) \rangle = P(\langle x, y \rangle).$$

**Problem**

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When $\theta = \pi/3$, the problem is known as a kissing number problem: what is the maximal number $\tau_d$ of non-overlapping unit balls touching a given unit ball in dimension d?

The kissing number is known in dimensions 2, 3, 4, 8, 24: $\tau_2 = 6$, $\tau_3 = 12$ [Schütte and van der Waerden, 1953], $\tau_4 = 24$ [Musin, 2003], $\tau_8 = 240$ and $\tau_{24} = 196560$ [Odlyzko and Sloane, Levenshtein, 1979].

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Asymptotic bound $\tau_d \leq 2^{0.401n(1+o(1))}$ [Kabatiansky and Levenshtein, 1978].

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**Theorem**
Let $P(t) = \alpha_1 G_{i_1}(t) + \ldots + \alpha_l G_{i_l}(t)$, where $\alpha_i > 0$ and $\sum \alpha_i = 1$, and $H = h_{i_1} + \ldots + h_{i_l}$. Then $M_d(I) \leq M_H(P(I))$.

**Lemma**

For $c > 0$, $M_d([-1, -c]) \leq 1/c + 1$.

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**Theorem (Delsarte's method)**

Let $P(t)$ be a non-negative linear combination of Gegenbauer polynomials, $P(1) = 1$, and $P(I) \subseteq [-1, -c]$ for $c > 0$. Then $M_d(I) \leq 1/c + 1$.

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$\tau_8 \leq 240$, $\tau_{24} \leq 196560$, the asymptotic bound of Kabatiansky and Levenshtein are proven by using Delsarte's method with the right choice of P.

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$M_d([-1, 0)) = d + 1.$

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Any $d+2$ points in $\mathbb{R}^d$ can be split into two sets whose convex hulls intersect.

**Proof of the Davenport-Hajós bound.**

Let v be a common point of two convex hulls from the Radon theorem. If all scalar products of unit vectors are negative then writing v as a positive combination of both sets, $v \cdot v < 0$. □

**Theorem**

Let $P(t) = \alpha_1 G_{i_1}(t) + \ldots + \alpha_l G_{i_l}(t)$, where $\alpha_i > 0$ and $\sum \alpha_i = 1$, and $H = h_{i_1} + \ldots + h_{i_l}$. If $P(I) \subseteq [-1, 0)$ then $M_d(I) \leq H + 1$.

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**Theorem (Orthoplex bound, Conway, Hardin, and Sloane, 1996)**

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The bound is sharp in a sense that for several configurations with more than $\binom{d+1}{2}$ points, scalar products are in $[-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}]$.

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Take $P(t) = \frac{1}{2}G_1(t) + \frac{1}{2}G_2(t) = \text{const} \cdot (t+1)(dt-1)$ and perturb it. Then $H = h_1 + h_2$ and $M_d([-1, \frac{1}{d})) \leq H + 1 = \frac{d(d+3)}{2}$. □

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Take $P(t) = \frac{2\sqrt{d}}{(d+2)(\sqrt{d}+1)}G_1(t) + \frac{1}{\sqrt{d}+1}G_2(t) + \frac{d\sqrt{d}}{(d+2)(\sqrt{d}+1)}G_3(t) = \text{const} \cdot (t + \frac{1}{\sqrt{d}})^2(t - \frac{1}{\sqrt{d}})$ and perturb it. Then $H = h_1 + h_2 + h_3$. Therefore, $M_d([-1, \frac{1}{\sqrt{d}})) \le H + 1 = \frac{d(d+1)(d+5)}{6}$. □

**Problem**

For a given potential $F$ and size $N$, find the minimum energy $E(d, N, F) = \min_X E_F(X)$ for $E_F(X) = \sum_{x,y \in X} F(\langle x, y \rangle)$, over all sets of points $X \subset \mathbb{S}^{d-1}$, $|X| = N$.

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**Proof.**

Let X be the minimizing set for $E(d, N, F(P))$. Then $E(H, N, F) \leq E_F(\phi_P(X)) = E(d, N, F(P))$. □

**Theorem (G.-Park, 2020)**

For $p \in [1, 2\log\frac{2m+1}{2m} / \log\frac{m+1}{m}]$, $F(t) = |t|^p$, and $1 \leq m \leq d$, $E(d, d+m, F) = 2m$.

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**Conjecture (Park, 2019)**

Repeated orthonormal bases with N vectors are energy minimizers for $F(t) = |t|^p$, $p \in [1, p(N)]$, for any $N \geq d$ and $p(N) \to 2$ when $N \to \infty$.

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**Proof.**

Take $P(t) = \frac{dt+1}{d+1}$. Use $\phi_P$ and repeated orthonormal bases. $\square$

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**Question**

How can we get nice configurations via mapping to the space of spherical harmonics?

For a positive integer t, a finite set $X \subset \mathbb{S}^{d-1}$ is called a spherical t-design if

$$\int_{\mathbb{S}^{d-1}} f(x) d\mu(x) = \frac{1}{|X|} \sum_{v \in X} f(v)$$

holds for all polynomials f of degree $\leq t$.

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A finite set $X \subset \mathbb{S}^{d-1}$ is called a (unit norm) tight frame if the above condition holds for all homogeneous polynomials of degree 2.

Tight frames $\leftrightarrow$ Antipodal 3-designs $\leftrightarrow$ Projective 1-designs

**Theorem (G., 2020)**

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**Lemma (Sidel'nikov; Venkov; Benedetto and Fickus)**

$X$ is a tight frame in $\mathbb{S}^{d-1}$ if and only if $\frac{1}{|X|^2}\sum_{x,y\in X}\langle x, y\rangle^2 = \frac{1}{d}$.

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**Lemma (Sidel'nikov; Venkov; Benedetto and Fickus)**

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**Proof of the theorem.**

$$\frac{1}{|X|^2}\sum_{x,y\in X} G_l(\langle x, y\rangle)^2 = \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} G_l(\langle x, y\rangle)^2 d\mu(x)d\mu(y) = \frac{1}{h_l}.$$

□

Let X be a 2k-design in $\mathbb{S}^{d-1}$. Let $P(t) = \alpha_1 G_{i_1}(t) + \ldots + \alpha_l G_{i_l}(t)$, where $\alpha_i > 0$, $\sum \alpha_i = 1$, $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k\}$ and $H = h_{i_1} + \ldots + h_{i_l}$. Then

$$\frac{1}{|X|^2} \sum_{x,y \in X} P(\langle x, y \rangle)^2 = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} P(\langle x, y \rangle)^2 d\mu(x) d\mu(y) =$$

$$= \frac{\alpha_1^2}{h_{i_1}} + \ldots + \frac{\alpha_l^2}{h_{i_l}} \geq \frac{(\alpha_1 + \ldots + \alpha_l)^2}{h_{i_1} + \ldots + h_{i_l}} = \frac{1}{H}$$

and the equality holds when $\alpha_j = \frac{h_{i_j}}{H}$.

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and the equality holds when $\alpha_j = \frac{h_{i_j}}{H}$.

**Theorem (G., 2020)**

If X is a 2k-design in $\mathbb{S}^{d-1}$ and $\{i_1, \ldots, i_l\}$ is a subset of $\{1, \ldots, k\}$ then $\phi_P(X)$ is a tight frame in $\mathbb{S}^{H-1}$, where $H = h_{i_1} + \ldots + h_{i_l}$ and $P(t) = \frac{h_{i_1}}{H} G_{i_1}(t) + \ldots \frac{h_{i_l}}{H} G_{i_l}(t)$.

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**Proof of the theorem.**

For $|X| = N$, $\sum_{x,y \in X} P(\langle x, y \rangle) \leq 1.23N$ and $\geq 0.09465869N^2$ so $N \leq 1.23/0.09465869 \approx 12.99405263$ □

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Case 1. There is one point in $C$. Then

$$\sum_{y \in X} P(\langle x, y \rangle) \leq P(1) + \max_{t \in [-1, -1/\sqrt{2}]} P(t) \leq 1.23.$$

Case 2. There are two points $y, z$ in $C$. To maximize the sum of values of $P$, $-x$ should lie on the geodesic between $y$ and $z$ and the angular distance between $y$ and $z$ should be $\pi/3$. Denoting $\langle x, y\rangle = t$, we find $\langle x, z\rangle = \alpha(t) = \frac{1}{2}t - \frac{\sqrt{3}}{2}\sqrt{1-t^2}$ and $t \in I = [-\cos\pi/12, -1/\sqrt{2}]$. Then

$$\sum_{y\in X} P(\langle x, y\rangle) \leq P(1) + \max_{t\in I}(P(t) + P(\alpha(t))) \leq 1.23.$$

Case 3. There are three points $y, z, w$ in $C$. To maximize the sum of values of $P$, the points $y, z, w$ should form a regular triangle with the sides of length $\pi/3$. Rotating the triangle with respect to the point furthest from $x$, we can increase the sum. The rotation stops either when one of the points reaches the boundary of $C$, or there are two points that are in the same distance from $x$. In the former case, we are left with two points in $C$. In the latter case, if $\langle x, y\rangle = \langle x, z\rangle = t$ then $\langle x, w\rangle = \beta(t) = \frac{2}{3}t - \frac{2}{3}\sqrt{\frac{3}{2} - 2t^2}$ and $t \in J = [-\frac{\sqrt{2}}{4} - \frac{1}{2}, -\sqrt{\frac{2}{3}}]$. Then

$$\sum_{y \in X} P(\langle x, y\rangle) \leq P(1) + \max_{t \in J}(2P(t) + P(\beta(t))) \leq 1.23.$$

THANK YOU!