The spherical cap discrepancy of HEALPix points

joint work with Julian Hofstadler and Michelle Mastrianni

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DOCTORAL PROGRAM
DISCRETE MATHEMATICS
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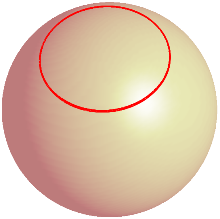
The Spherical Cap Discrepancy

A *spherical cap* with center $w \in \mathbb{S}^2$ and height $t \in (-1, 1)$ is given by the set

$$C(w,t) = \{x \in \mathbb{S}^2 \ : \ \langle x, w \rangle \geq t\}.$$



Local spherical cap discrepancy
Let $Z_N = \{z_1, \ldots, z_N\} \subset \mathbb{S}^2$.

$$\mathcal{D}_{sc}^{C(w,t)}(Z_N) = \left| \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{C(w,t)}(z_j) - \sigma(C(w,t)) \right|.$$

Spherical cap discrepancy

$$\mathcal{D}_{sc}(Z_N) = \sup_{w \in \mathbb{S}^2} \sup_{-1 \leq t \leq 1} \mathcal{D}_{sc}^{C(w,t)}(Z_N).$$

Integration error

Let $P_N = \{x_1, \dots, x_N\} \subset [0,1]^d$, and f be a function of bounded Variation*, then

$$\left| \frac{1}{N} \sum_{j=1}^{N} f(x_j) - \int f \, \mathrm{d}x \right| \leq \mathcal{D}(P_N) \mathcal{V}(f).$$

Funktionen von beschränkter Variation in der Theorie der Gleichverteilung, Annali di Matematica Pura ed Applicata 54 - Hlawka (1961).

Beck showed that point sets $\omega_N^\star$ exist with constants $c, C > 0$ independent of N, such that

$$cN^{-3/4} \leq \mathcal{D}_{sc}(\omega_N^\star) \leq CN^{-3/4}\sqrt{\log N}.$$

Sums of distances between points on a spherean application of the theory of irregularities of distribution to discrete geometry, Mathematika 31 (1) - Beck (1984).

Some known spherical cap discrepancy

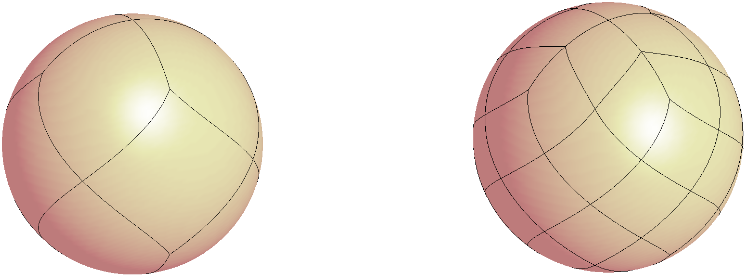
- i.i.d. Random points are of order $N^{-1/2}$,
- Fibonacci points F_N satisfy $\mathcal{D}_{sc}(F_N) \leq N^{-1/2}$
- for a certain Diamond ensemble D_N, $\mathcal{D}_{sc}(D_N) \sim N^{-1/2}$.

Point Sets on the Sphere $\mathbb{S}^2$ *with Small Spherical Cap Discrepancy*. Discrete Comput Geom 48 - Aistleitner, Brauchart, Dick (2012).

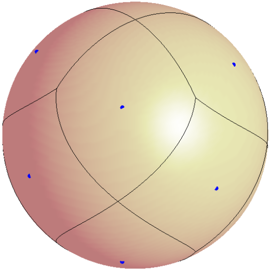
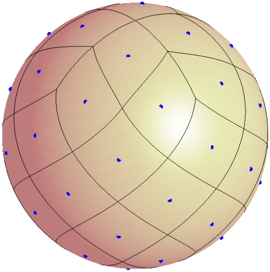
Spherical Cap Discrepancy of the Diamond Ensemble, Discrete Comput Geom - Etayo (2021).

The HEALPix Lattice

Hierarchical, Equal Area and iso-Latitude Pixelation.



HEALPix: A Framework for High-Resolution Discretization and Fast Analysis of Data Distributed on the Sphere, The Astrophysical Journal 622, pp. 759-771 - Górski, Hivon, Banday, Wandelt, Hansen, Reinecke and Bartelmann (2005).

The point set H_N

Placing points at centers. In total $N = 12 * 4^{\ell}$ many points at level ℓ.

Pixel boundaries in the equatorial belt

Let $L = 2^\ell$ and $j \in \{0, 1, \ldots, 4L-1\}$

$$\begin{array}{rcl} \phi_j^\ell : I_j^\ell & \to & \left[0, \frac{\pi}{2}\right] \\ \phi & \mapsto & \phi - \frac{j}{L}\frac{\pi}{2} \end{array} \qquad \text{with} \qquad I_j^\ell := \left[\frac{j}{L}\frac{\pi}{2}, \frac{j+L}{L}\frac{\pi}{2}\right].$$

$$m_{j,\ell}^e \sim \cos(\theta) = \frac{2}{3} - \frac{8}{3\pi}\phi_j^\ell \qquad \text{and} \qquad p_{j,\ell}^e \sim \cos(\theta) = -\frac{2}{3} + \frac{8}{3\pi}\phi_j^\ell,$$

meaning $m_{j,\ell}^e = \big(\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)\big)$.

The projection map Γ

$$\begin{array}{rcl} \Gamma : \mathbb{S}^2 & \to & [-\frac{1}{2}, \frac{1}{2}] \times [0,2] \\ x & \mapsto & \Gamma(x) \end{array}$$

- If $|\cos(\theta)| \leq \frac{2}{3}$ and $\phi \in [0, 2\pi]$, then

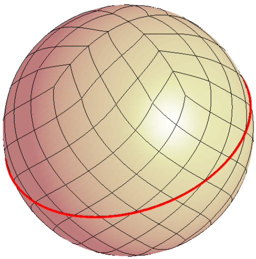
$$x \mapsto \Gamma(x) = \begin{pmatrix} \phi/\pi \\ 3/8 \cos(\theta) \end{pmatrix}.$$

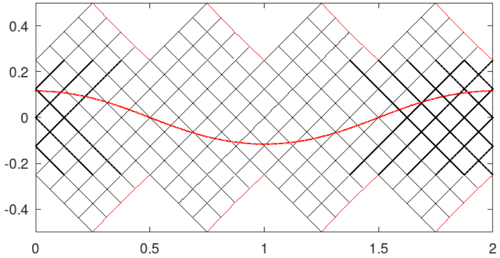
- If $\cos(\theta) > \frac{2}{3}$ and $\phi \in [0, 2\pi]$, then

$$x \mapsto \Gamma(x) = \frac{1}{\pi} \begin{pmatrix} \phi - \left(1 - \sqrt{1-\cos(\theta)}\sqrt{3}\right) \cdot (\phi \mod \frac{\pi}{2} - \frac{\pi}{4}) \\ \frac{\pi}{4}\left(2 - \sqrt{1-\cos(\theta)}\sqrt{3}\right) \end{pmatrix}.$$

In the equatorial belt we obtain with $\phi \in I_j^l$:

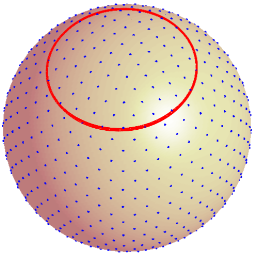
$$\Gamma(m_{j,l}^e) = \frac{1}{\pi}\begin{pmatrix} \phi \\ -\phi - \frac{j}{L}\frac{\pi}{2} + \frac{\pi}{4} \end{pmatrix} \quad \text{and} \quad \Gamma(p_{j,l}^e) = \frac{1}{\pi}\begin{pmatrix} \phi \\ \phi - \frac{\pi}{4} - \frac{j}{L}\frac{\pi}{2} \end{pmatrix}.$$

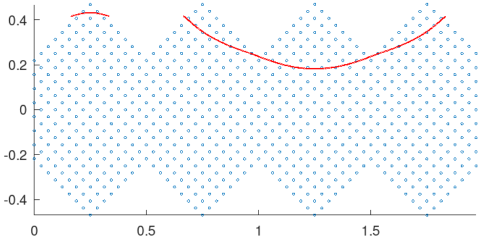




Given a base pixel B and an open set $A \subset B$, then

$$\frac{\mathrm{area}(A)}{\mathrm{area}(B)} = \frac{\mathrm{area}(\Gamma(A))}{\mathrm{area}(\Gamma(B))}.$$





Theorem (DF, Hofstadler, Mastrianni '21)

$$N^{-1/2} \leq \mathcal{D}_{sc}(H_N) \leq 1000\ N^{-1/2}.$$

Let $C = C(w,t)$ where $w \in \mathbb{S}^2$ and $t \in [0,1)$, then

$$\partial C = \Big\{ \gamma(\phi,\theta) \ : \ \sin(\theta)\sin(\theta_w)\cos(\phi - \phi_w) + \cos(\theta)\cos(\theta_w) = t \Big\}.$$

If $\sin(\theta_w) \neq 0$, then

$$\phi = \phi_w + \arccos\left(\frac{t - \cos(\theta)\cos(\theta_w)}{\sin(\theta)\sin(\theta_w)}\right) \text{ and/or}$$

$$\phi = \phi_w - \arccos\left(\frac{t - \cos(\theta)\cos(\theta_w)}{\sin(\theta)\sin(\theta_w)}\right).$$

We calculate the signed curvature κ of the planar curve $\Gamma(\partial C)$

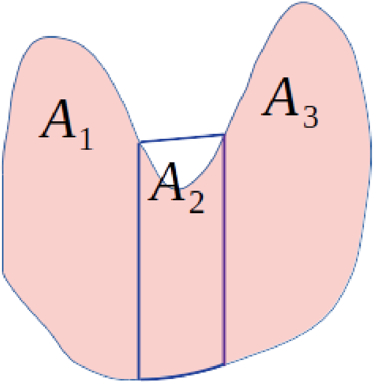
$$\kappa = \frac{x'y'' - y'x''}{\left((x')^2 + (y')^2\right)^{3/2}}$$

and find its zeros.

Pseudo-convex set

A set A (pink) is pseudo-convex if

1. $\exists\ A_1, \ldots, A_p$ convex sets,
2. $A_j \cap A_k = \emptyset$,
3. $A \subseteq A_1 \cup \cdots \cup A_p$,
4. either $A_j \subset A$ or $A_j \setminus A$ is convex.



Lemma (Aistleitner, Brauchart, Dick '12) For $P_N, A \subset [0,1]^2$ with A pseudo-convex,

$$\Big|\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_A(x_n) - \lambda(A)\Big| \leq (2p-q) J_N(P_N).$$

Isotropic discrepancy

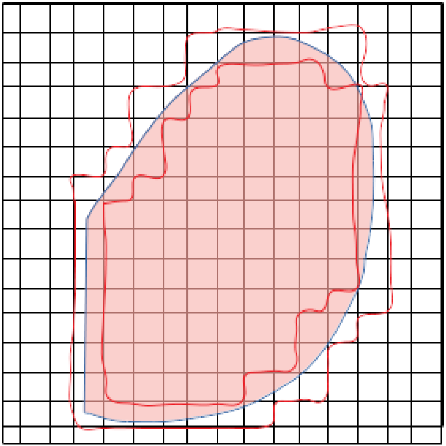
$$J_N(P_N) = \sup_{F \in \mathcal{F}} \Big| \frac{\#\{n : 1 \leq n \leq N, \mathbf{x}_n \in F\}}{N} - \text{area}(F) \Big|,$$

where $\mathcal{F}$ is the family of all convex subsets of $[0,1]^2$.

Isotropic discrepancy of $\Gamma(H_N)$

For a given convex set K,

$$\left|\frac{\#\{\mathbf{x}_n \in K\}}{N} - \text{area}(K)\right| \leq 4N^{-1/2}.$$



Thank you for your Time