Discrete logarithmic energy

Let $\omega_N = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$.

The **logarithmic energy** of ω_N:

$$\mathcal{E}_{\log}(\omega_N) = \sum_{i \neq j} \log \frac{1}{||x_i - x_j||}$$
$$= -\sum_{i \neq j} \log ||x_i - x_j||.$$

1) Limit case of the Riesz s-energy

$$\mathcal{E}_s(\omega_N) = \sum_{i \neq j} \frac{1}{||x_i - x_j||^s}$$

$$\mathcal{E}_{\log}(\omega_N) = \frac{d}{ds}\bigg|_{s=0} \mathcal{E}_s(\omega_N)$$
$$= -\sum_{i \neq j} \log \|x_i - x_j\|.$$

2) A facility location problem.

Minimizing the logarithmic energy

$$\min_{\omega_N \subset \mathbb{S}^2} \sum_{i \neq j} \log \frac{1}{||x_i - x_j||} = \max_{\omega_N \subset \mathbb{S}^2} \prod_{i \neq j} ||x_i - x_j||$$

1) Facility location problem!
(Beltrán, 14')

2) Smale 7th problem!

Smale 7th problem

$$m_N = \min_{\omega_N \subset \mathbb{S}^2} \mathcal{E}_{\log}(\omega_N).$$

Give a set of N points $\omega_N \subset \mathbb{S}^2$ such that

$$|\mathcal{E}_{\log}(\omega_N) - m_N| \leq c \log(N),$$

for a universal constant c.

$$m_N = \min_{\omega_N \subset \mathbb{S}^2} \mathcal{E}_{\log}(\omega_N).$$

1) 1923, Fekete

$$\lim_{N\to\infty} \frac{m_N}{N^2} = \frac{1}{2} - \log(2)$$

$$= \int_{x,y\in\mathbb{S}^2} \log\left(\frac{1}{||x-y||}\right) dxdy.$$

$$\Rightarrow m_N = \left(\frac{1}{2} - \log(2)\right) N^2 + o(N^2).$$

2) 1988, Elkies and Lang

$$m_N \geq \left(\frac{1}{2} - \log(2)\right) N^2 - \frac{1}{2} N \log(N) + O(N).$$

3) 1989, Wagner

$$m_N \geq \left(\frac{1}{2} - \log(2)\right) N^2 - \frac{1}{2} N \log(N) + CN.$$

4) Rakhmanov, Saff and Zhou; Dubickas and Brauchart; Brauchart, Hardin and Saff.

5) 2018, Bétermin and Sandier

$$m_N = \left(\frac{1}{2} - \log(2)\right) N^2 - \frac{1}{2} N \log(N) + cN + o(N).$$

Conjecture (Bétermin and Sandier; Brauchart, Hardin and Saff):

$$c = 2\log(2) + \frac{1}{2}\log\left(\frac{2}{3}\right) + 3\log\left(\frac{\sqrt{\pi}}{\Gamma(1/3)}\right) = -0.0556053...$$

Equiv. to the Cohn-Kumar conjecture. (2019, Petrache and Serfaty).

Conjecture (2007, Cohn and Kumar) In dimension $d = 2, 8, resp. 24$, the lattice Λ_0 is universally minimizing in the sense that it minimizes E_p among all possible point configurations of density 1 for all $p's$ that are completely monotone functions of the squared distance.

Theorem (2019, Cohn, Kumar, Miller, Radchenko and Viazovska). The Cohn-Kumar conjecture is true in dimensions $d = 8$ and 24.

Smale 7th problem

$$m_N = \min_{\omega_N \subset \mathbb{S}^2} \mathcal{E}_{\log}(\omega_N).$$

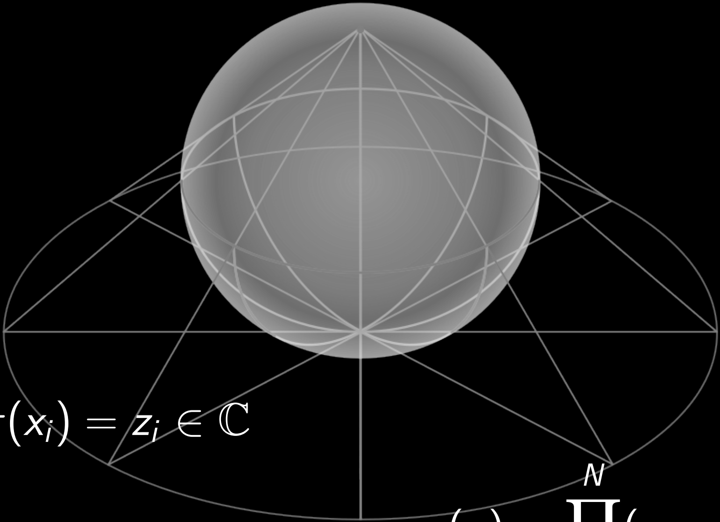
Give a set of N points $\omega_N \subset \mathbb{S}^2$ such that

$$|\mathcal{E}_{\log}(\omega_N) - m_N| \leq c \log(N),$$

for a universal constant c.

Motivation: Shub and Smale, 1993.

$$\omega_N = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$$



$$\pi(x_i) = z_i \in \mathbb{C}$$

$$p_N(x) = \prod_{j=1}^{N} (x - z_i)$$

The condition number of a polynomial

$$f(x) = a_0 + a_1 x + \ldots + a_N x^N$$

at its root z is given by

$$\mu(f, z) = \frac{\sqrt{N}(1 + |z|^2)^{\frac{N-2}{2}}}{f'(z)} ||f||.$$

$$\mu(f) = \max_{z:f(z)=0} \mu(f, z).$$

Problem (1993, Shub and Smale). Give explicitly a sequence of polynomials $(p_N)_N$ such that

$$\mu(p_N) \leq N^a.$$

with $a \in \mathbb{N}$ a fix number.

Theorem (1993, Shub and Smale). Elliptic Fekete polynomials are well conditioned.

Theorem (2020, Beltrán, E., Marzo and Ortega-Cerdà). We solve Shub and Smale problem.

Theorem 1.7. *For all* $N \geq 1$ *let* $M, r_1, \ldots, r_M$ *be an admissible set of integers. Define the the parallel heights*

$$h_j = 1 - \frac{2}{N}\sum_{k=1}^{j-1} r_k - \frac{r_j}{N}, \quad H_j = h_j - \frac{r_j}{N},$$

for $1 \leq j \leq M-1$, *and let* $r_j = 6s_j + rem_j$ *with* $rem_j \in \{0, \ldots, 5\}$ *for* $2 \leq j \leq M$. *Then there exist a constant* $C > 0$ *such that the polynomials* $P_N(z) = P_N^{(1)}(z)P_N^{(2)}(z)P_N^{(3)}(z)P_N^{(4)}(z)$ *with*

$$\begin{aligned}
P_N^{(1)}(z) &= \left(z^{4s_M + rem_M} - 1\right)\left(z^{r_1} - \rho(h_1)^{r_1}\right)\left(z^{r_1} - 1/\rho(h_1)^{r_1}\right), \\
P_N^{(2)}(z) &= \left(z^{s_2} - \rho(H_1)^{s_2}\right)\left(z^{s_2} - 1/\rho(H_1)^{s_2}\right), \\
P_N^{(3)}(z) &= \prod_{j=2}^{M-1}\left(z^{4s_j + rem_j} - \rho(h_j)^{4s_j + rem_j}\right)\left(z^{4s_j + rem_j} - 1/\rho(h_j)^{4s_j + rem_j}\right), \\
P_N^{(4)}(z) &= \prod_{j=2}^{M-1}\left(z^{s_j + s_{j+1}} - \rho(H_j)^{s_j + s_{j+1}}\right)\left(z^{s_j + s_{j+1}} - 1/\rho(H_j)^{s_j + s_{j+1}}\right),
\end{aligned}$$

where if $s_2 = 0$ *or if* $s_j + s_{j+1} = 0$ *the corresponding term is removed from the product and* $\rho(x) = \sqrt{(1-x)/(1+x)}$ *satisfy*

$$\mu_{\text{norm}}(P_N) \leq C\sqrt{N}.$$

Theorem (1993, Shub and Smale). Elliptic Fekete polynomials are well conditioned.

$$\mathcal{E}_{\log}(\omega_N) \leq m_N + c \log N$$
$$\Rightarrow \mu(p_N(x)) \leq \sqrt{N^{1+c}(N+1)}$$

Theorem (2019, E.). The roots of well conditioned polynomials proyected to the sphere have small logarithmic energy.

$$\mu(p_N(x)) \leq C\sqrt{N} \Rightarrow \mathcal{E}_{\log}(\omega_N) \leq m_N + cN,$$

where c a constant depending only on C.

$$\mathcal{E}_{\log}(\omega_N) = \sum_{i=1}^{N} \log(\mu(p_N, z_i)) + N \log \left(\frac{\prod_{i=1}^{N} \sqrt{1 + |z_i|^2}}{||p_N||} \right) - \log(2) N^2 - \frac{N \log(N)}{2} + \log(2) N.$$

$$\frac{\prod_{i=1} \sqrt{1+|z_i|^2}}{||p_N||} = \frac{\prod_{i=1} ||x - z_i||}{\left|\left|\prod_{i=1}^{N}(x - z_i)\right|\right|}$$

Theorem (2019, E.). Given a set of complex points $z_1, \ldots, z_N$, we have

$$\prod_{i=1}^{N} ||x - z_i|| \leq \sqrt{\frac{e^N}{N+1}} \left|\left|\prod_{i=1}^{N}(x - z_i)\right|\right|,$$

where $|| * ||$ is the Bombieri-Weyl norm and the bound is sharp up to a constant.

Bombieri's inequality (1990, Beauzamy, Bombieri, Enflo and Montgomery). Let P, Q be homogeneous polynomials of degrees m, n respectively. Then

$$||PQ|| \geq \sqrt{\frac{m!n!}{(m+n)!}}||P||\,||Q||,$$

where $||*||$ is the Bombieri-Weyl norm and the inequality is sharp.

$$\Rightarrow \prod_{i=1}^{N} ||x - z_i|| \leq \sqrt{N!}\left|\left|\prod_{i=1}^{N}(x - z_i)\right|\right|.$$

Conjecture (2019, E.).

$$\prod_{i=1}^{N} \|x - z_i\| \leq K_N \sqrt{\frac{e^N}{N+1}} \left\| \prod_{i=1}^{N} (x - z_i) \right\|$$

$$\Rightarrow K_N = A + o(1) \forall N \in \mathbb{N}?$$