High-dimensional sphere packing and the modular bootstrap

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Sphere packing

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In dimension 8 and 24 the optimal configurations are the E_8 root lattice and the Leech lattice; optimality proved using the Cohn-Elkies linear programming bound and Viazovska's (quasi)modular forms

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Up to subexponential factors this is still the best known lower bound

The spinless modular bootstrap

The partition functions of certain conformal field theories are of the form

$$\mathcal{Z}(\tau) = \sum_{\Delta} d_\Delta \chi_\Delta(\tau)$$

where

$$\chi_\Delta(\tau) = \frac{e^{2\pi i \tau \Delta}}{\eta(\tau)^{2c}} \quad \text{and} \quad \mathcal{Z}(-1/\tau) = \mathcal{Z}(\tau)$$

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How large can the the spectral gap be?

The parameter here is c and the variables are

$$\Delta_0 < \Delta_1 < \Delta_2 < \dots \qquad \text{and} \qquad d_{\Delta_k} \in \mathbb{N}_1$$

where $\Delta_0 = 0$ and $d_0 = 1$

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Such ω proves the upper bound $\Delta_1 \leq \Delta_{\text{gap}}$

Uncertainty principle

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They write the optimization over ω as an optimization problem over functions $f\colon \mathbb{R}^{2c} \to \mathbb{R}$ not identically zero with

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This type of uncertainty principle introduced by Bourgain, Clozel, and Kahane, and recently studied by Cohn and Gonçalves in connection to the sphere packing problem

Cohn-Elkies bound

Theorem

Let $h\colon \mathbb{R}^d \to \mathbb{R}$ be an integrable, continuous, radial function such that $\widehat{h}$ is integrable, and let r be a positive real number. If

- $h(0) = \widehat{h}(0) = 1$,
- $h(x) \leq 0$ whenever $|x| \geq r$, and
- $\widehat{h} \geq 0$,

then every sphere packing in $\mathbb{R}^d$ has density at most the volume of a sphere of radius $r/2$ in $\mathbb{R}^d$

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If h is a function satisfying the above hypotheses, then $f = \widehat{h} - h$ satisfies the hypotheses on the previous slide with $c = d/2$:

$$\widehat{f} = -f \quad \text{and} \quad f(x) \geq 0 \quad \text{whenever} \quad |x| \geq \frac{\Delta_{\text{gap}}^2}{2}$$

Numerical optimization

Consider f as a function of $\Delta = |x|^2/2$ and write

$$f(\Delta) = \sum_{k=1}^{2N} \alpha_k f_k(\Delta), \quad \text{where} \quad f_k(\Delta) = L_{2k-1}^{(c-1)}(4\pi\Delta)e^{-2\pi\Delta},$$

$L_{2k-1}^{(c-1)}$ is the Laguerre polynomial with parameter $c-1$ of degree $2k-1$, so that f_k (as a function of x) is an eigenfunction for the Fourier transform with eigenvalue -1

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Checking whether f satisfies the inequality constraint

$$f(\Delta) \geq 0 \quad \text{for} \quad \Delta \geq \Delta_{\text{gap}}$$

is a semidefinite program, but this is computationally expensive for large N

Numerical optimization

Instead of using semidefinite programming, fix Δ_1 and maximize $f(0)$ over α_k for $1 \leq k \leq 2N$ and Δ_n for $2 \leq n \leq N$, subject to

$$f(\Delta_1) = 0 \text{ and}$$
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Afkhami-Jeddi, Hartman, and Tajdini recently improved this method by dualizing the problem, which gives

$$f_k(0) + \sum_{n=1}^{N} d_n f_k(\Delta_n) = 0 \text{ for } 1 \le k \le 2N.$$

These are $2N$ polynomial equations in $2N$ variables, and a solution can be found efficiently by Newton's method

Orthogonal polynomials

$$w^{a,b}(t) = \frac{\Gamma(a+b+d-1)}{2^{a+b+d-2}\Gamma(a+\frac{d-1}{2})\Gamma(b+\frac{d-1}{2})}(1-t)^a(1+t)^b(1-t^2)^{(d-3)/2},$$

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Define $Q_i^{a,b}$ with $\deg(Q_i^{a,b}) = i$ and positive leading coefficient by

$$\int_{-1}^{1} Q_i^{a,b}(t) Q_j^{a,b}(t) w^{a,b}(t)\,dt = \delta_{i,j}$$

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Jacobi polynomials with parameters $(d-3)/2+a$ and $(d-3)/2+b$

Orthogonal polynomials

For a continuous function $f\colon [-1,1] \to \mathbb{R}$,

$$f_0 := \int_{-1}^{1} f(t) w^{0,0}(t)\, dt = \int_{S^{d-1}} f(\langle x, e \rangle)\, d\mu(x),$$

where $e \in S^{d-1}$ is an arbitrary point and μ is the surface measure on the sphere S^{d-1}, normalized so that $\mu(S^{d-1}) = 1$

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Addition formula: $\frac{Q_i(\langle x,y\rangle)}{Q_i(1)} = \frac{1}{r_i} \sum_{j=1}^{r_i} v_{i,j}(x) v_{i,j}(y)$

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Schoenberg: These polynomials generate all positive type functions

Delsarte-Goethals-Seidel bound

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Theorem

Let $f\colon [-1, 1] \to \mathbb{R}$ be a continuous function and $s \in [-1, 1]$. If f is of positive type as a zonal function on S^{d-1}, $f(t) \leq 0$ for $t \in [-1, s]$, and $f_0 \neq 0$, then $A(d, s)$ is at most $f(1)/f_0$.

Levenshtein's universal bound

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Define

$$f^{(s)}(t) = \begin{cases} (t-s)\big(K_{k-1}^{1,0}(t,s)\big)^2 & \text{if } t_{k-1}^{1,1} \le s < t_k^{1,0}\text{, and} \\ (t+1)(t-s)\big(K_{k-1}^{1,1}(t,s)\big)^2 & \text{if } t_k^{1,0} \le s < t_k^{1,1}, \end{cases}$$

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where

$$K_{k-1}^{1,0}(t,s) = \sum_{i=0}^{k-1} Q_i^{1,0}(t) Q_i^{1,0}(s).$$

That $(t-s)\big(K_{k-1}^{1,0}(t,s)\big)$ is of positive type follows using the Christoffel-Darboux formula

$$(t-s)K_{k-1}^{1,0}(t,s) = (c_{k-1}/c_k)\big(Q_k^{1,0}(t)Q_{k-1}^{1,0}(s) - Q_k^{1,0}(s)Q_{k-1}^{1,0}(t)\big)$$

The Kabatianskii-Levenshtein bound

$$\Delta_{\mathbb{R}^d} \leq \min_{\pi/3 \leq \theta \leq \pi} \sin^d(\theta/2) A(d, \cos\theta) = \min_{-1 \leq s \leq 1/2} \left(\frac{1-s}{2}\right)^{d/2} A(d, s)$$

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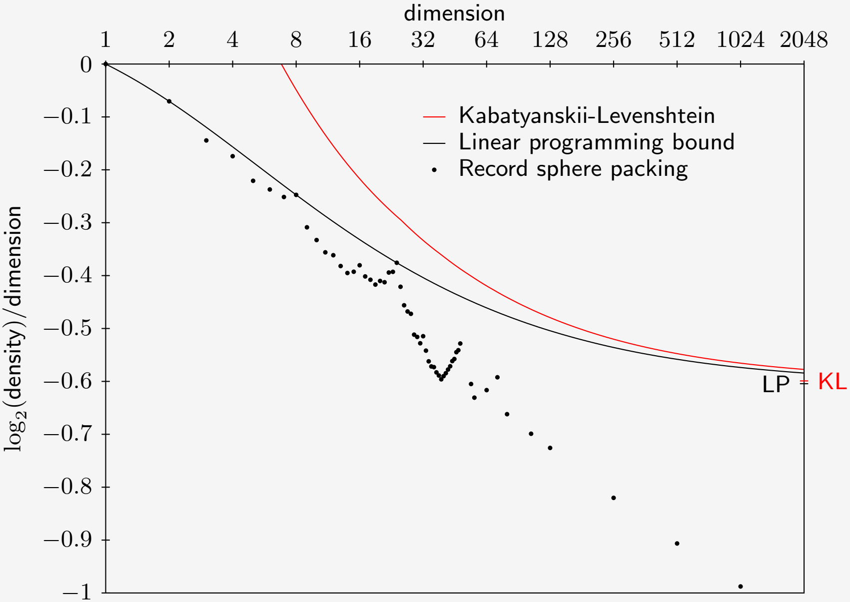
This gives the asymptotic upper bound

$$\Delta_{\mathbb{R}^d} \leq 2^{-(0.59905576+o(1))d}$$

Asymptotic upper bounds



Asymptotic upper bounds

d	LP	KL	KL − LP	Difference	Ratio
1	0.00000	0.00000	0.00000		
2	−0.07049	0.29248	0.36297		
4	−0.15665	0.17511	0.33176	0.03122	0.23
8	−0.24737	−0.04879	0.19858	0.13318	1.59
16	−0.33192	−0.21692	0.11501	0.08357	1.87
32	−0.40382	−0.33342	0.07040	0.04461	1.55
64	−0.46101	−0.41947	0.04154	0.02885	1.73
128	−0.50432	−0.47947	0.02485	0.01669	1.82
256	−0.53589	−0.52023	0.01566	0.00919	1.87
512	−0.55824	−0.54749	0.01075	0.00491	1.90
1024	−0.57370	−0.56553	0.00816	0.00259	1.92
2048	−0.58418	−0.57737	0.00682	0.00135	1.92
4096	−0.59120	−0.58508	0.00611	0.00070	
∞	?	−0.59906	?		

Asymptotic upper bounds

Conjecture

The Cohn-Elkies bound proves

$$\Delta_{\mathbb{R}^n} \leq 2^{-(\lambda+o(1))d}$$

for some $0.604 < \lambda < 0.605$ when the auxiliary function is fully optimized

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Equivalently

$$\lim_{d\to\infty} \frac{\mathsf{A}_-(d)}{\sqrt{d}} = \frac{1}{\pi},$$

where $\mathsf{A}_-(d)$ denotes the optimal radius for the uncertainty principle mentioned before

Implied kissing numbers

Suppose h is an optimal auxiliary function for the Cohn-Elkies bound

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By a scaling argument by Cohn and Miller, the average kissing number $d_1^{\mathrm{LP}}(d/2)$ of any sphere packing attaining the Cohn-Elkies bound must be equal to

$$-\frac{d}{rh'(r)}$$

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$$-\frac{d}{rh'(r)}$$

This is called the implied kissing number of the bound

Implied kissing numbers

$\log_2(\cdot)/d$ of the Cohn-Elkies bound and the implied kissing number:

d	LP	IK	1 + LP − IK
1	0.00000	1.00000	0.00000
2	-0.07049	1.29248	-0.36297
4	-0.15665	1.18103	-0.33768
8	-0.24737	0.98836	-0.23573
16	-0.33192	0.81510	-0.14702
32	-0.40382	0.68279	-0.08661
64	-0.46101	0.58837	-0.04937
128	-0.50432	0.52325	-0.02757
256	-0.53589	0.47929	-0.01518
512	-0.55824	0.45003	-0.00827
1024	-0.57370	0.43077	-0.00447
2048	-0.58418	0.41822	-0.00240
4096	-0.59120	0.41009	-0.00128
∞	-0.6044	0.3956	0.00000

Implied kissing numbers

The modular invariance of the partition function implies

$$\sum_{\Delta} d_{\Delta} e^{-\beta\Delta} \sim (2\pi/\beta)^c \quad \text{as} \quad \beta \to 0$$

Implied kissing numbers

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Using a tauberian theorem this implies

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$\Delta_2^{\mathrm{LP}}(c) - \Delta_1^{\mathrm{LP}}(c)$ is sublinear in c, from which it follows that the implied kissing number is $2^{d+o(d)}$ times the Cohn-Elkies bound for sphere packing

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We show that whenever we can prove a bound on the average kissing number which is better than the implied kissing number, a theorem by L, Oliveira, Vallentin can be used to improve the Cohn-Elkies bound

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Could it be true that the Delsarte-Goethals-Seidel bound for the kissing number problem can be used to strenghten the Cohn-Elkies bound in all dimensions except $d = 1, 2, 8, 24$?

Implied kissing numbers

Ratio of the implied kissing number to the Delsarte-Goethals-Seidel kissing number bound

Implied kissing numbers

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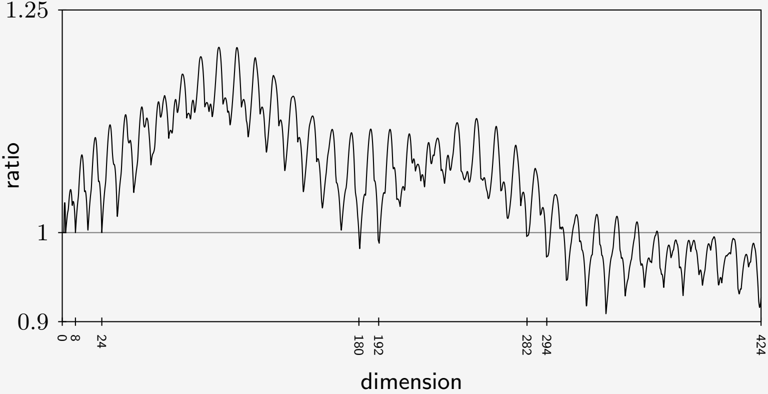

Thank you!