Lubotzky-Phillips-Sarnak points on a sphere

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This is an expository talk based on work of Lubotzky-Phillips-Sarnak

Lubotzky-Phillips-Sarnak 1986/87

- Points on sphere constructed from

$$\begin{array}{ccc} p & = & a^2 + b^2 + c^2 + d^2 \\ \text{prime} & = & \text{sum of integer squares} \end{array}$$

- Why 4 squares for 2d sphere?

$$(a, b, c, d) \quad \Rightarrow \quad \text{quaternion} \quad \Rightarrow \quad \text{rotation of } S^2$$

- Proof that points are well distributed uses theorem of Deligne (Ramanujan conjecture on Fourier coefficients of holomorphic cuspforms)

- Fact that their properties are best possible uses theorem of Kesten (random walk on groups)

Quadrature: $\sum$ vs $\int$

- Operator: given finite set of rotations S, define

$$Tf(x) = \sum_S f(S^{-1}x)$$

self-adjoint on L^2 if the set also contains S^{-1} for each of its S

- Largest eigenvalue: for $f = 1$ get

$$Tf = 2\ell f$$

where $2\ell = \#$ of rotations

- Want Tf small when $\int f = 0$

No such luck on torus

- Rotations $\Rightarrow$ translations

$$x \mapsto x \pm a_i \qquad i = 1, \ldots, \ell$$

- Constant function gives highest eigenvalue 2ℓ as before
- But there are exponentials with eigenvalue arbitrarily close to 2ℓ
- $f(x) = \exp(2\pi\sqrt{-1}\nu \cdot x)$ well-defined on torus $\mathbb{R}^n/\mathbb{Z}^n$ for $\nu \in \mathbb{Z}^n$

$$Tf = \lambda_\nu f \quad \text{where} \quad \lambda_\nu = 2\sum_{i=1}^{\ell} \cos 2\pi\nu \cdot a_i \approx 2\ell$$

provided $\nu \cdot a_i$ are all close to integers

λ_1

- Next largest $|$eigenvalue$|$ of T, after $2\ell = \lambda_0$

$$\lambda_1 = \sup \|Tf\|$$

over f with $\int f = 0$ and $\int |f|^2 = 1$

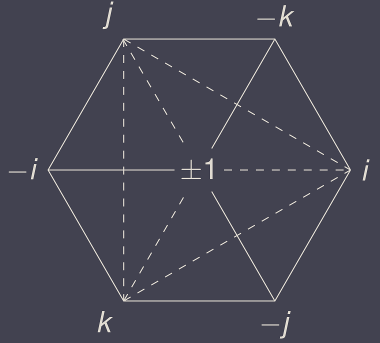
- λ_1 controls mean-square error in the approximation

$$\int f \approx \frac{1}{2\ell} \sum_S f(Sx)$$

over different choices of x

- Want spectral gap: λ_1 as far from 2ℓ as possible

Quaternions and rotations


j
−k
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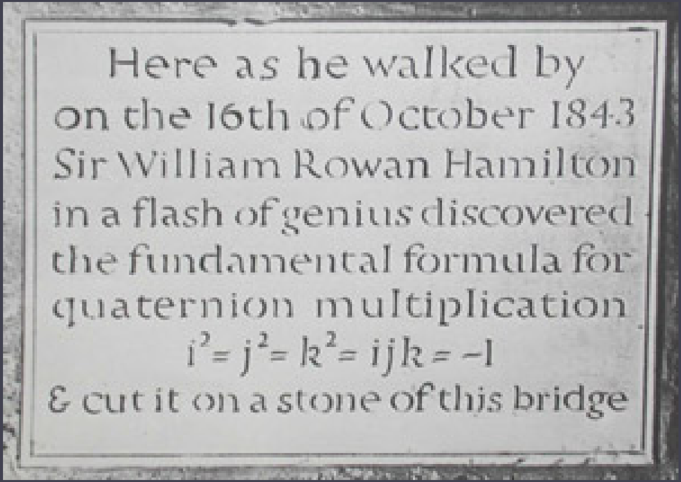

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
i² = j² = k² = ijk = −1
& cut it on a stone of this bridge

Rotations

- $q = w + xi + yj + zk$ has a conjugate $\bar{q} = w - xi - yj - zk$
- Quaternion norm
$$N(q) = q\bar{q} = w^2 + x^2 + y^2 + z^2$$
Crucially $N(q_1 q_2) = N(q_1)N(q_2)$ so
$$N(q^{-1}rq) = N(r)$$
- If r has "real part" $w = 0$ then so does $q^{-1}rq$
$N(r) = x^2 + y^2 + z^2$ is also preserved
- $\Rightarrow$ Each quaternion q defines a rotation of the sphere

Which rotation is it?

- Polar form

$$q = s(\cos\theta + u\sin\theta)$$

where s is a scalar and $\quad u = xi + yj + zk \quad$ with $x^2 + y^2 + z^2 = 1$
Note that $u^2 = -1$ for any such "unit quaternion"

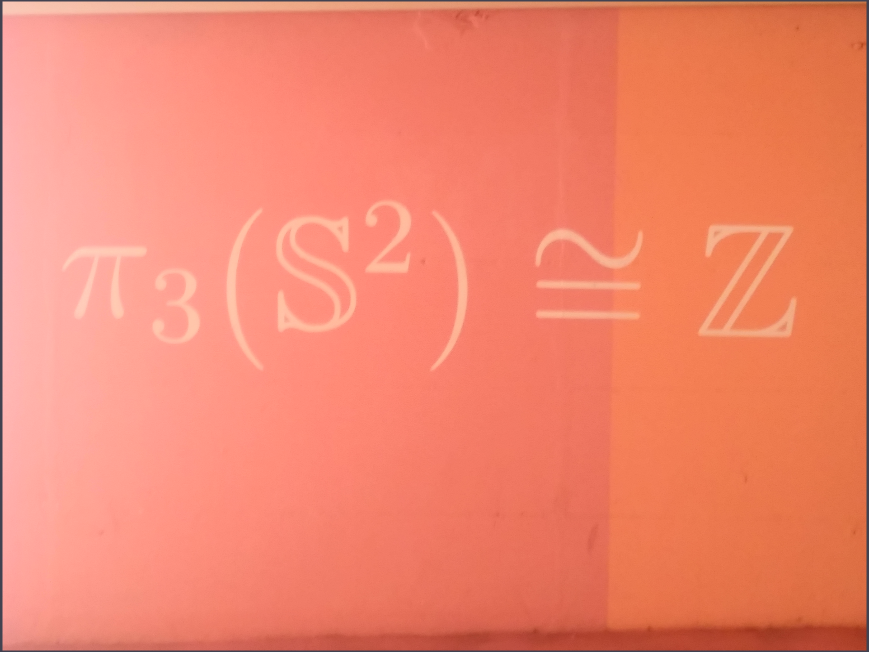
- Then $r \mapsto q^{-1}rq$ is a rotation by angle 2θ around axis u

- Compare with matrix form of the same rotation

$$\begin{pmatrix} u & u_1^\perp & u_2^\perp \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} u & u_1^\perp & u_2^\perp \end{pmatrix}^\top$$

Example: $p = 5$

- $5 = N(q)$ for any of the six quaternions $1 \pm 2i, 1 \pm 2j, 1 + \pm 2k$
- Polar form $\sqrt{5}(\cos\theta + u \sin\theta)$ where $\cos\theta = 1/\sqrt{5}$ and $u = \pm i, j$, or k
- Corresponding rotations have orthogonal axes X, Y, or Z (counter)clockwise according to $\pm \quad (\implies S^{-1}$ always comes with $S)$
- Trig identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ determines angle $\arccos(-3/5)$ in degrees: roughly 126.86989764584402129685561255909341066


$\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$

Sums of four squares

$8(p+1)$ ways to write p as a sum of four squares

- $5 = 1^2 + 2^2 + 0^2 + 0^2$

$8(p+1) = 48 = 6\times$	$2\times$	2^2
two places out of four for 0	choose 1 vs 2	choose sign for non-zero coordinates

- $7 = 2^2 + 1^2 + 1^2 + 1^2$
  $8(p+1) = 64 = 4 \times 2^4$ four places for 2, then four signs to choose

- $11 = 0^2 + 1^2 + 1^2 + 3^2$
  $8(p+1) = 12 \times 2^3$ four places for 0, three left for 3, three signs to choose

- First really non-unique case: $p = 13$ with $8(p+1) = 8 \times 6 + 8 \times 8$

$$13 = 3^2 + 2^2 + 0^2 + 0^2 \quad \text{48 like this}$$
$$= 1^2 + 2^2 + 2^2 + 2^2 \quad \text{64 like this}$$

Jacobi's 4-square theorem

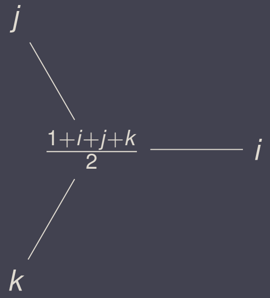
- $8(p+1)$ ways to write prime p as sum of four squares
- For composite n, replace $p+1$ by sum of divisors of n excluding multiples of 4
- Only $p+1$ ways of a standard form:
  $w + xi + yj + zk$ where w has different parity from the others

$$\text{e.g.}\quad 5 = 1^2 + 2^2 + 0^2 + 0^2 \quad \text{with one odd, others even}$$
$$7 = 2^2 + 1^2 + 1^2 + 1^2 \quad \text{with one even, others odd}$$

- For each prime p, get $p+1$ quaternions of norm p such that
  $q \bmod 2$ is either 1 (if $p \equiv 1 \bmod 4$) or $i+j+k$ (if $p \equiv 3 \bmod 4$)
  with real part $w > 0$
- Assume $p \equiv 1 \bmod 4$ from now on

Hurwitz quaternions

- $w + xi + yj + zk$ with coefficients either all integers, or all half-integers (gives D_4 instead of $\mathbb{Z}^4$)
- enables division with remainder for quaternions and a form of unique factorization
- clarifies why Jacobi excludes multiples of 4: need to separate integer solutions from half-integer



Why does this achieve $\lambda_1 = 2\sqrt{p}$?

- Amplify difference between λ_1 and $2\sqrt{p}$ by iterating many times

$$T_n f(\zeta) = \frac{1}{2} \sum_{\alpha} f(\alpha\zeta)$$

sum over quaternions $\alpha \equiv 1 \mod 2$ with norm $N(\alpha) = n$

- Lemma: T_{p^m} is a polynomial in T_p

$$T_{p^m} = U_m(T_p) \qquad \text{where} \qquad p^{m/2} \frac{\sin(m+1)\theta}{\sin\theta} = U_m(2\sqrt{p}\cos\theta)$$

- Write eigenvalue as $\lambda = 2\sqrt{p}\cos\theta$
  Want to show θ is real so $\lambda_1 \leq 2\sqrt{p}$

- If $T_p u = \lambda u$, with eigenvalue written $\lambda = 2\sqrt{p}\cos\theta$

$$T_{p^m} u = p^{m/2} \frac{\sin{(m+1)\theta}}{\sin\theta} u$$

- Input from Deligne: for any harmonic u and any point ζ on the sphere

$$|T_{p^m} u(\zeta)| \ll_\varepsilon p^{m/2+\varepsilon m}$$

constant depends on ε, u, ζ BUT NOT on m

- Consequence: As $m \to \infty$

$$p^{\varepsilon m} \gg \left|\frac{\sin{(m+1)\theta}}{\sin\theta}\right| \approx e^{m|\text{Imag }\theta|}$$

exp rate $\varepsilon \log p$ exp rate $|\text{Imag }\theta|$

Only possible if θ is real, since ε can be arbitrarily small

From p to p^m

- Every integral quaternion with $N(\beta) = p^m$ has a unique representation

$$\beta = \pm p^s (\text{product of } t \text{ quaternions of norm } p)$$

  where $2s + t = m$, the factors are in "standard form" $\alpha \equiv 1 \bmod 2$ and the product is reduced (no cancellations $\alpha\alpha^{-1}$ allowed)

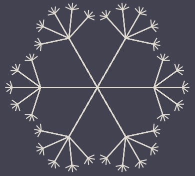
-

$$T_{p^m} f(\zeta) = \sum_{N(\beta)=p^m} f(\beta\zeta) = \sum_{s \le m/2} \sum_{w} f(w\zeta)$$

- inner sum over shell at radius $t = m - 2s$ in $(p+1)$-regular tree
- Recurrence

$$G_m(x) = xG_{m-1}(x) - pG_{m-2}(x)$$

  solved by linear combo of exponentials; initial values match $\sin{(m+1)\theta}/\sin\theta$



Theta series

- Generating function for the terms $T_n u$ we want to estimate
- Given spherical harmonic u and point ζ, let

$$\theta(z) = \sum_{\alpha} N(\alpha)^m u(\alpha\zeta) \exp\left(2\pi\sqrt{-1}N(\alpha)z/16\right)$$

  sum over integral quaternions α with $\alpha \equiv 2 \bmod 4$
  converges for $\Im(z) > 0$

- Collect terms:

$$\theta(z) = \sum_{\nu=1}^{\infty}\left(\nu^m \sum_{\alpha} u(\alpha\zeta)\right) \exp(2\pi\sqrt{-1}\nu z/16)$$

  inner sum over α with $N(\alpha) = \nu$ and $\alpha \equiv 2 \bmod 4$

$|\sum_\alpha u(\alpha\zeta)| \ll \nu^{1/2+\varepsilon}$ where $N(\alpha) = \nu$, $\alpha \equiv 2 \bmod 4$

- If the spharmonic u is non-constant, then
  θ is a holomorphic cuspform of weight $2 + 2m$ for $\Gamma(4)$, meaning roughly:
  - θ is periodic under certain translations of z
  - further properties derived from Poisson sum
  - θ is not too large as $\Im(z) \to 0, \infty$ (would fail for $u = 1$)
- Deligne's theorem: for any holomorphic cuspform of weight k

$$\nu^{\text{th}}\text{coefficient} \ll_\varepsilon \nu^{k/2-1/2+\varepsilon}$$

- For θ, the coefficients are $\nu^m \sum_\alpha u(\alpha\zeta)$ and $k = 2 + 2m$

Back to p^m

- Apply Deligne's theorem with $\nu = 4p^m$
- Change (quaternion) variables to $\beta = \alpha/2$
- Get

$$\left| \sum_{\substack{\beta \equiv 1 \bmod 2 \\ N(\beta) = p^m}} u(\beta\zeta) \right| \ll_\varepsilon (p^m)^{1/2+\varepsilon}$$

which is the input we needed earlier:

$$|T_{p^m} u(\zeta)| \ll p^{m/2} p^{\varepsilon m}$$

Why can’t we do even better?

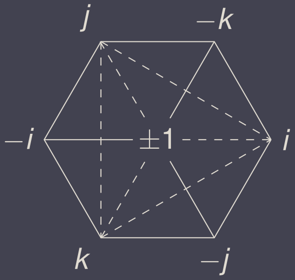
Cayley graph

- Given a group G with generating set S
  Assume $S = \{\gamma_1^{\pm 1}, \dots, \gamma_\ell^{\pm 1}\}$ is symmetric (and finite)
- The vertices of the Cayley graph are the elements of G
  Edges connect g to sg for each generator s from the given set S
- Adjacency operator on a graph:

$$Tf(x) = \sum_{y \sim x} f(y)$$

  sum over all neighbours of the point x

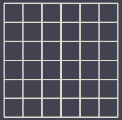
A Cayley graph we have met



- Group G of order 8
- Generating set S consists of i, j, k and inverses

More typically, G would be infinite

e.g. $\mathbb{Z}^2$ with generators $(1, 0)$ and $(0, 1)$



$\lambda_1 \geq 2\sqrt{2\ell - 1}$ for any set of 2ℓ rotations

- In particular $\lambda_1 \geq 2\sqrt{p}$ for the $p+1$ rotations from $p = a^2 + b^2 + c^2 + d^2$
- Theorem (Kesten 1959)
$$2\sqrt{2\ell - 1} \leq \|T\| \leq 2\ell$$
where T is the adjacency operator of a Cayley graph on 2ℓ generators
- $\|T\| = 2\sqrt{2\ell - 1}$ if and only if G is the free group on those generators

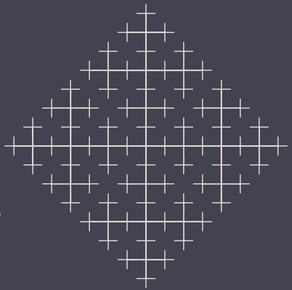
Banach-Tarski

- Quaternions $1 + 2i$ and $1 + 2j$ from $p = 5$
  generate a free group of rotations
  (freedom guaranteed by Kesten; possible to check directly)

- CHOOSE a set R of representatives from each orbit

- Every point of S^2 lies in wR for some word w in the generators
  $a = 1 + 2\mathrm{i}, \quad b = 1 + 2j, \quad$ and their inverses

- Partition sphere based on first letter of w




That's all Folks!

For more information

- Lubotzky-Phillips-Sarnak,
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- Conway and Smith,
  On Quaternions and Octonions (2003)
- Deligne, La conjecture de Weil I (1974)
  Pub Math IHES, et II (1980)
- Kesten, Symmetric random walks on groups (1959) Transactions of the AMS