Weighted $\mathbb{L}^2$-Norms of Gegenbauer Polynomials — and more!

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Based on

joint work with

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on

Weighted $\mathbb{L}^2$-norms of Gegenbauer polynomials

https://arxiv.org/abs/2103.08303

Outline

Origins

Consider

$$\int_I (p_n(x))^2 \, w(x) \, \mathrm{d}\, x$$

for orthogonal polynomials $p_0, \ldots$ on interval I not necessarily orthogonal w.r.t. weight w.

- Extends work of Damir Ferizović (https://arxiv.org/abs/1909.08121);
- inspired by Carlos Beltran and Damir Ferizovic. *Approximation to uniform distribution in SO(3).* Constr. Approx., 52(2):283–311, 2020. (https://arxiv.org/abs/1901.10840v1)
- physics papers;

Gegenbauer Polynomials $\mathrm{C}_n^{(\lambda)}$

- **Generating function relation:**

$$\sum_{n=0}^{\infty} \mathrm{C}_n^{(\lambda)}(x)\, z^n = \frac{1}{(1-2xz+z^2)^\lambda}.$$

- ...classical orthogonal polynomials w.r.t. weight function $\left(1-x^2\right)^{\lambda-\frac{1}{2}}$ on $[-1,1]$:

$$\int_{-1}^{1} \mathrm{C}_n^{(\lambda)}(x)\, \mathrm{C}_m^{(\lambda)}(x) \left(1-x^2\right)^{\lambda-\frac{1}{2}} \mathrm{d}\, x = \delta_{n,m}\, h_n^{(\lambda)}.$$

- **Standard normalisation:**

$$\mathrm{C}_n^{(\lambda)}(1) = \frac{(2\lambda)_n}{n!} = \frac{1}{\Gamma(2\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)} \sim \frac{1}{\Gamma(2\lambda)}\, n^{2\lambda-1}.$$

Orthogonality relation yields

$$
\begin{aligned}
\int_{-1}^{1} \left(\mathrm{C}_n^{(\lambda)}(x) \right)^2 (1 - x^2)^{\lambda - \frac{1}{2}} \,\mathrm{d}\, x
&= \frac{\sqrt{\pi}\,\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)} \frac{\lambda}{n + \lambda} \frac{(2\lambda)_n}{n!} \\
&= \frac{\sqrt{\pi}\,\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\,\Gamma(2\lambda)} \frac{1}{n + \lambda} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + 1)} \\
&\sim \frac{\pi}{2^{2\lambda - 1}\,(\Gamma(\lambda))^2}\, n^{2\lambda - 2}.
\end{aligned}
$$

What if the weight function is “WRONG” ?

Quantity of Interest

Definition

Let $\lambda > 0$ and $\alpha, \beta > -1$. For $n \in \mathbb{N}$,

$$I_n^{(\lambda;\alpha,\beta)} := \int_{-1}^{1} \left(\mathrm{C}_n^{(\lambda)}(x) \right)^2 (1-x)^{\alpha} (1+x)^{\beta} \, \mathrm{d}\, x.$$

Remark

Since $\mathrm{C}_n^{(\lambda)}(-x) = (-1)^n \, \mathrm{C}_n^{(\lambda)}(x)$,

$$I_n^{(\lambda;\beta,\alpha)} = I_n^{(\lambda;\alpha,\beta)} \qquad \text{and} \qquad \text{w.l.o.g.: } -1 < \alpha \leq \beta.$$

Explicit Formulas

- **Pochhammer symbol**

$$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}.$$

- **Classical hypergeometric functions**

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n \, n!} z^n$$

for $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{C}$ and $p \leq q+1$.

These power series allow for an analytic continuation to the slit complex plane $\mathbb{C} \setminus [1, \infty)$.

Theorem

$$I_n^{(\lambda;\alpha,\beta)} = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \times \left(\frac{(2\lambda)_n}{n!} \right)^2 {}_5F_4\left(\left. \begin{matrix} -n, n+2\lambda, \lambda, \alpha+1, \beta+1 \\ 2\lambda, \lambda+\frac{1}{2}, \frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2} \end{matrix} \right| 1 \right).$$

Proof.

Follows from J. Sanchez-Ruiz, 2001

$$\left(\mathrm{C}_n^{(\lambda)}(x) \right)^2 = \left(\frac{(2\lambda)_n}{n!} \right)^2 {}_3F_2\left(\left. \begin{matrix} -n, n+2\lambda, \lambda \\ 2\lambda, \lambda+\frac{1}{2} \end{matrix} \right| 1-x^2 \right)$$

and

$$\int_{-1}^{1} (1-x^2)^k \, (1-x)^\alpha \, (1+x)^\beta \, \mathrm{d}\, x = 2^{2k+\alpha+\beta+1} \frac{\Gamma(k+\alpha+1)\,\Gamma(k+\beta+1)}{\Gamma(2k+\alpha+\beta+2)}.$$

□

Remark

For $\beta = \alpha = \mu - \frac{1}{2}$, the 1-balanced ${}_5F_4$-hypergeometric polynomial reduces to

$$
{}_4F_3\left(\begin{matrix} -n, n+2\lambda, \lambda, \mu+\frac{1}{2} \\ 2\lambda, \lambda+\frac{1}{2}, \mu+1 \end{matrix} \middle| 1\right).
$$

For $\mu = \lambda$, the ${}_4F_3$ becomes

$$
{}_3F_2\left(\begin{matrix} -n, n+2\lambda, \lambda \\ 2\lambda, \lambda+1 \end{matrix} \middle| 1\right),
$$

hence can be computed by the Pfaff-Saalschütz theorem as

$$
\frac{(\lambda)_n(-n)_n}{(2\lambda)_n(-n-\lambda)_n} = \frac{\lambda}{n+\lambda}\frac{n!}{(2\lambda)_n}.
$$

Explicit Formulas, Gegenbauer Weight

Set: $J_n^{(\lambda;\mu)} := I_n^{(\lambda;\mu-\frac{1}{2},\mu-\frac{1}{2})} = \int_{-1}^{1} \left(\mathrm{C}_n^{(\lambda)}(x) \right)^2 (1-x^2)^{\mu-\frac{1}{2}} \, \mathrm{d}\, x.$

Theorem

Let $\mu > -\frac{1}{2}$ *and* $\lambda > 0$. *Then*

$$J_n^{(\lambda;\mu)} = \frac{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu+1)} \left(\frac{(2\lambda)_n}{n!} \right)^2 \underbrace{{}_4F_3\left(\begin{matrix} -n, n+2\lambda, \lambda, \mu+\frac{1}{2} \\ 2\lambda, \lambda+\frac{1}{2}, \mu+1 \end{matrix} \middle| 1 \right)}_{\text{alternating sum}};$$

alternatively, we have (limits if $\mu = 0$*)*

$$J_n^{(\lambda;\mu)} = \frac{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu+1)} \underbrace{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(\lambda)_{n-k}^2 (\lambda-\mu)_k^2}{(\mu+1)_{n-k}^2 (k!)^2} \frac{n+\mu-2k}{\mu} \frac{(2\mu)_{n-2k}}{(n-2k)!}}_{\text{positive sum}}.$$

Remark

If $\mu - \lambda = k \in \mathbb{N}$, then

$$J_n^{(\lambda;\lambda+k)} = \frac{2\pi}{4^{\lambda+k}\Gamma(\lambda)^2} \sum_{\ell=0}^{\min(k,\lfloor \frac{n}{2} \rfloor)} \binom{k}{\ell}^2 \left(\frac{\Gamma(n-\ell+\lambda)}{\Gamma(n+k-\ell+1+\lambda)} \right)^2 \times (n-2\ell+k+\lambda) \frac{\Gamma(n+2k-2\ell+2\lambda)}{\Gamma(n-2\ell+1)}$$

and, e.g.,

$$J_n^{(\lambda;\lambda+k)} = \frac{2\pi}{4^{\lambda+k}\Gamma(\lambda)^2} \binom{2k}{k} n^{2\lambda-2} + \mathcal{O}(n^{2\lambda-3}).$$

Connection Formula and Special Cases

Theorem

Let $\alpha > -1$ *and* $\lambda > 0$. *Then for* $\beta - \alpha = k \in \mathbb{N}$,

$$I_n^{(\lambda;\alpha,\alpha+2m+\eta)} = \sum_{\ell=0}^{m} (-1)^\ell b_\ell \, J_n^{(\lambda;\ell+\alpha+\frac{1}{2})},$$

where

$$b_\ell = \binom{m}{\ell} \frac{(m+\eta)_{m-\ell}}{\left(\frac{1}{2}+\eta\right)_{m-\ell}} \times \begin{cases} 1, & \eta = 0 \\ 2m+1, & \eta = 1. \end{cases}$$

Generating Functions

Theorem

$$\overbrace{\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} I_n^{(\lambda;\alpha,\beta)} z^n}^{\mathcal{I}^{(\lambda;\alpha,\beta)}(z)}$$

$$= 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$$

$$\times \frac{1}{(1-z)^{2\lambda}} \, {}_4F_3\left(\begin{matrix} \lambda, \lambda, \alpha+1, \beta+1 \\ 2\lambda, \frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2} \end{matrix} \middle| -\frac{4z}{(1-z)^2} \right).$$

The ${}_4F_3$ is "$\frac{1}{2}$-balanced".

Proof.

Interchanging order of summation in

$$\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} I_n^{(\lambda;\alpha,\beta)} z^n = A \sum_{\ell=0}^{\infty} \sum_{n=\ell}^{\infty} c_\ell \frac{(2\lambda)_{n+\ell}}{(n-\ell)!} z^n$$

and using

$$\sum_{n=\ell}^{\infty} \frac{(2\lambda)_{n+\ell}}{(n-\ell)!} z^n = \frac{1}{(1-z)^{2\lambda}} (\lambda)_\ell \left(\lambda + \frac{1}{2}\right)_\ell \left(\frac{4z}{(1-z)^2}\right)^\ell.$$

□

Remark

For $\beta = \alpha = \mu - \frac{1}{2}$,

$$\overbrace{\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} \underbrace{I_n^{(\lambda;\mu-\frac{1}{2},\mu-\frac{1}{2})}}_{J_n^{(\lambda;\mu)}} z^n}^{\mathcal{J}^{(\lambda;\mu)}(z)} = \frac{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu+1)} \frac{1}{(1-z)^{2\lambda}} \, {}_3\mathrm{F}_2\left(\begin{matrix} \lambda, \lambda, \mu+\frac{1}{2} \\ 2\lambda, \mu+1 \end{matrix} \middle| -\frac{4z}{(1-z)^2} \right).$$

The same rhs is obtained for $\alpha = \mu + \frac{1}{2}$ and $\beta = \mu - \frac{1}{2}$, since

$$\int_{-1}^{1} \left(\mathrm{C}_n^{(\lambda)}(x)\right)^2 (1-x)(1-x^2)^{\mu-\frac{1}{2}} \,\mathrm{d}\,x = \int_{-1}^{1} \left(\mathrm{C}_n^{(\lambda)}(x)\right)^2 (1-x^2)^{\mu-\frac{1}{2}} \,\mathrm{d}\,x.$$

Asymptotic Results

Asymptotics – Main Term, Jacobi Weights

Theorem

Let $-1 < \alpha < \beta$ *and* $\lambda > 0$ *be real numbers. Then (with* $\alpha_0 := \lambda - 1$*)*

$$I_n^{(\lambda;\alpha,\beta)} = \begin{cases} \frac{2^{\alpha+\beta+2-4\lambda}\Gamma(\alpha+1-\lambda)\Gamma(\beta+1-\lambda)}{\Gamma(\lambda)^2\Gamma(\alpha+\beta+2-2\lambda)} n^{2\lambda-2} + \mathcal{O}(n^\eta) & \alpha > \alpha_0, \\ \frac{2^{\beta+2-3\lambda}}{\Gamma(\lambda)^2} n^{2\lambda-2}\left(\log n - A(\lambda,\beta)\right) + \mathcal{O}\left(n^{2\lambda-3}\log n\right) + \mathcal{O}\left(n^{4\lambda-2\beta-5}\right) & \alpha = \alpha_0, \\ \frac{2^{\beta-3\alpha-3}\,\Gamma(\alpha+1)\,\Gamma(\lambda-\alpha-1)^2}{\Gamma(\lambda)^2\,\Gamma(2\lambda-\alpha-1)} n^{4\lambda-2\alpha-4} + \mathcal{O}(n^\eta) & \alpha < \alpha_0, \end{cases}$$

where

$$\eta = \begin{cases} \max(2\lambda-3, 4\lambda-2\alpha-4) & \alpha > \alpha_0, \\ \max(2\lambda-2, 4\lambda-2\alpha-5, 4\lambda-2\beta-4) & \alpha < \alpha_0, \end{cases}$$

and

$$A(\lambda,\beta) = \frac{1}{2}\left(\gamma - 4\log 2 + \psi(\beta+1-\lambda) + 2\,\psi(\lambda)\right).$$

Asymptotics – Main Term, Gegenbauer Weights

Recall: $J_n^{(\lambda;\mu)} = I_n^{(\lambda;\mu-\frac{1}{2},\mu-\frac{1}{2})} = \int_{-1}^{1} \left(\mathrm{C}_n^{(\lambda)}(x)\right)^2 (1-x^2)^{\mu-\frac{1}{2}} \,\mathrm{d}\, x.$

Theorem

Let $\mu > -\frac{1}{2}$ *and* $\lambda > 0$. *Then (with* $\mu_0 := \lambda - \frac{1}{2}$*; i.e.,* $\alpha_0 = \lambda - 1$*)*

$$J_n^{(\lambda;\mu)} = \begin{cases} \dfrac{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2}-\lambda)}{2^{2\lambda-1}\Gamma(\lambda)^2\Gamma(\mu+1-\lambda)}\, n^{2\lambda-2} + \mathcal{O}(n^\eta) & \mu > \mu_0, \\ \dfrac{1}{2^{2\lambda-2}\Gamma(\lambda)^2}\, n^{2\lambda-2}\left(\log n + 2\log 2 - \psi(\lambda)\right) + \mathcal{O}\left(n^{2\lambda-3}\log n\right) & \mu = \mu_0, \\ \dfrac{\sqrt{\pi}\,\Gamma(\lambda-\mu-\frac{1}{2})\Gamma(\mu+\frac{1}{2})}{2^{2\lambda-1}\Gamma(\lambda)^2\,\Gamma(\lambda-\mu)\Gamma(2\lambda-\mu-\frac{1}{2})} n^{4\lambda-2\mu-3} + \mathcal{O}(n^\eta) & \mu < \mu_0, \end{cases}$$

where

$$\eta = \begin{cases} \max(2\lambda-3, 4\lambda-2\mu-3) & \mu > \mu_0, \\ \max(2\lambda-2, 4\lambda-2\mu-4) & \mu < \mu_0. \end{cases}$$

Complete Asymptotics

Cf. Next Section.

Singularity Analysis

SIAM J. Discrete Math., 3(2), 216–240. (25 pages)

Singularity Analysis of Generating Functions

Philippe Flajolet and Andrew Odlyzko

https://doi.org/10.1137/0403019

This work presents a class of methods by which one can translate, on a term-by-term basis, an asymptotic expansion of a function around a dominant singularity into a corresponding asymptotic expansion for the Taylor coefficients of the function. This approach is based on

Permalink: `https://doi.org/10.1137/0403019`

- **Main advantage over the classical method of Darboux:** . . . able to obtain asymptotic expressions for the coefficents of generating functions in the case that the coefficients tend to 0.

Singularity analysis needs information on the behaviour of the analytic continuation to a region of the form

$$\Delta_{\varepsilon,\phi} = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 + \varepsilon, |\arg(1 - z)| < \phi \right\}$$

for some $\pi > \phi > \frac{\pi}{2}$ (if radius of convergence is 1).

Theorem

Assume that, with the sole exception of the singularity $z = 1$, $f(z)$ is analytic in $\Delta_{\varepsilon,\phi}$ for some $\varepsilon > 0$ and $\phi > \frac{\pi}{2}$. Assume further that as z tends to 1 in $\Delta_{\varepsilon,\phi}$,

$$f(z) = \mathcal{O}(|1 - z|^{\alpha})$$

for some real number α. Then the n-th Taylor coefficient of $f(z)$ satisfies

$$f_n = [z^n]f(z) = \mathcal{O}(n^{-\alpha-1}).$$

Consequently, a local expansion around $z = 1$ of the generating function

$$f(z) = \sum_{k=0}^{K} a_k (1 - z)^{\alpha_k} + \mathcal{O}\left(|1 - z|^{\beta}\right)$$

for $\alpha_0 < \alpha_1 < \cdots < \alpha_K < \beta$ translates into an asymptotic relation for the coefficients

$$f_n = \sum_{k=0}^{K} a_k \binom{n - \alpha_k - 1}{n} + \mathcal{O}(n^{-\beta-1}).$$

Each of the binomial coefficients has an asymptotic expansion in terms of powers of n:

$$\binom{n - \alpha_k - 1}{n} = \frac{1}{\Gamma(-\alpha_k)} \frac{\Gamma(n - \alpha_k)}{\Gamma(n + 1)} = \frac{n^{-\alpha_k - 1}}{\Gamma(-\alpha_k)} \left\{1 + \mathcal{O}\left(\frac{1}{n}\right)\right\}.$$

Mellin-Barnes Formula, Jacobi Weights

$$
\begin{aligned}
&\mathcal{I}^{(\lambda;\alpha,\beta)}(z) \\
&\quad = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \\
&\qquad\qquad \times \frac{1}{(1-z)^{2\lambda}} \, {}_4F_3\!\left(\begin{matrix} \lambda, \lambda, \alpha+1, \beta+1 \\ 2\lambda, \frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2} \end{matrix} \,\middle|\, -\frac{4z}{(1-z)^2} \right) \\
&\quad = \frac{2^{\alpha+\beta+1}\Gamma(2\lambda)}{\Gamma(\lambda)^2(1-z)^{2\lambda}} \\
&\qquad \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s+\lambda)^2\Gamma(s+\alpha+1)\Gamma(s+\beta+1)\Gamma(-s)}{\Gamma(s+2\lambda)\Gamma(2s+\alpha+\beta+2)} \left(\frac{16z}{(1-z)^2} \right)^s \mathrm{d}\, s,
\end{aligned}
$$

where the contour of integration is taken along the imaginary axis encircling $s = 0$ in the left half-plane such that the poles at $s = -\lambda$, $s = -\alpha - 1$, and $s = -\beta - 1$ are to the left of the contour.

Derivation of the Asymptotics – short version

Moving the contour to the left and collecting the residues in the **generic case** yields (after reordering terms)

$$\begin{aligned}\mathcal{I}^{(\lambda;\alpha,\beta)}(z) &= \sum_{m=0}^{\infty} A_m^{(\lambda;\alpha,\beta)} \, (1-z)^{m+2+2\alpha-2\lambda} \\ &\quad + \sum_{m=0}^{\infty} B_m^{(\lambda;\alpha,\beta)} \, (1-z)^{m+2+2\beta-2\lambda} \\ &\quad + \sum_{m=0}^{\infty} D_m^{(\lambda;\alpha,\beta)} \, (1-z)^{m} \log \frac{1}{1-z} \\ &\quad + \text{power series in } (1-z).\end{aligned}$$

Singularity Analysis gives asymptotic series for the coefficients

$$I_n^{(\lambda;\alpha,\beta)} \sim \frac{(2\lambda)_n}{n!}\left(\sum_{m=0}^{\infty} D_m (-1)^m \frac{m!}{n(n-1)\cdots(n-m)} + \sum_{m=0}^{\infty} A_m \binom{n+2\lambda-2\alpha-3-m}{n} + \sum_{m=0}^{\infty} B_m \binom{n+2\lambda-2\beta-3-m}{n}\right).$$

$A_m = A_m^{(\lambda;\alpha,\beta)}$, $B_m = B_m^{(\lambda;\alpha,\beta)}$, and $D_m = D_m^{(\lambda;\alpha,\beta)}$ are explicit.

Mellin-Barnes Formula, Gegenbauer Weights

Similarly,

$$\mathcal{J}^{(\lambda;\mu)}(z) = \frac{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu+1)} \frac{1}{(1-z)^{2\lambda}} {}_3F_2\left(\begin{matrix}\lambda, \lambda, \mu+\frac{1}{2} \\ 2\lambda, \mu+1\end{matrix} \middle| -\frac{4z}{(1-z)^2}\right)$$

$$= \frac{\sqrt{\pi}\,\Gamma(2\lambda)}{\Gamma(\lambda)^2(1-z)^{2\lambda}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s+\lambda)^2\Gamma(s+\mu+\frac{1}{2})\Gamma(-s)}{\Gamma(s+2\lambda)\Gamma(s+\mu+1)} \left(\frac{4z}{(1-z)^2}\right)^s \mathrm{d}\, s,$$

where the contour is chosen as before, this time leaving $s = -\lambda$ and $s = -\mu - \frac{1}{2}$ to the left of the contour.

Asymptotics, Generic Case

Collecting residues at the double poles $-\lambda - \ell$, $\ell \in \mathbb{N}_0$, gives

$$\frac{\Gamma(\lambda + \frac{1}{2})\,\Gamma(\frac{1}{2} + \mu - \lambda)}{\Gamma(\lambda)\,\Gamma(1 + \mu - \lambda)} \left(z^{-\lambda}\, {}_3F_2\!\left(\begin{matrix} 1 - \lambda, \lambda, \lambda - \mu \\ 1, \frac{1}{2} + \lambda - \mu \end{matrix} \,\middle|\, -\frac{(1-z)^2}{4z} \right) \log \frac{1}{1-z} \right.$$
$$\left. + \text{power series in } (1 - z) \right).$$

Collecting residues at the simple poles $-\mu - \frac{1}{2} - \ell$, $\ell \in \mathbb{N}_0$, gives

$$\frac{\Gamma(\lambda + \frac{1}{2})\,\Gamma(\lambda - \mu - \frac{1}{2})^2\,\Gamma(\mu + \frac{1}{2})}{4^{1+\mu-\lambda}\sqrt{\pi}\;\Gamma(\lambda)\,\Gamma(2\lambda - \mu - \frac{1}{2})}$$
$$\times \frac{(1-z)^{1+2\mu-2\lambda}}{z^{\mu+\frac{1}{2}}}\, {}_3F_2\!\left(\begin{matrix} \frac{1}{2}, \mu + \frac{1}{2}, \frac{3}{2} + \mu - 2\lambda \\ \frac{3}{2} + \mu - \lambda, \frac{3}{2} + \mu - \lambda \end{matrix} \,\middle|\, -\frac{(1-z)^2}{4z} \right).$$

Thus

$$\mathcal{J}^{(\lambda;\mu)}(z) = \sum_{m=0}^{\infty} D_m^{(\lambda;\mu)} (1-z)^m \log \frac{1}{1-z} + \sum_{m=0}^{\infty} A_m^{(\lambda;\mu)} (1-z)^{m+1+2\mu-2\lambda} + \text{power series in } (1-z).$$

Asymptotic analysis gives

$$J_n^{(\lambda;\mu)} = \frac{(2\lambda)_n}{n!} \left(\sum_{m=0}^{\infty} D_m^{(\lambda;\mu)} (-1)^m \frac{m!}{n(n-1)\cdots(n-m)} + \sum_{m=0}^{\infty} A_m^{(\lambda;\mu)} \binom{n+2\lambda-2\mu-2-m}{n} \right).$$

Special Case $\mu = \lambda - \frac{1}{2}$, Dominant Term

$$
\begin{aligned}
&\mathcal{J}^{(\lambda;\lambda-\frac{1}{2})}(z) \\
&\quad = \frac{\sqrt{\pi}\,\Gamma(2\lambda)}{\Gamma(\lambda)^2(1-z)^{2\lambda}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s+\lambda)^3\Gamma(-s)}{\Gamma(s+2\lambda)\Gamma(s+\lambda+\frac{1}{2})} \left(\frac{4z}{(1-z)^2}\right)^s \mathrm{d}\, s \\
&\quad = \frac{\Gamma(\lambda+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\lambda)} z^{-\lambda} \left(\gamma - \log 2 + \psi(\lambda) - \frac{1}{2}\log\frac{4z}{(1-z)^2}\right)^2 \\
&\qquad + \frac{\sqrt{\pi}\,\Gamma(2\lambda)}{\Gamma(\lambda)^2(1-z)^{2\lambda}} \underbrace{\frac{1}{2\pi i} \int_{-\lambda-1-i\infty}^{-\lambda-1+i\infty} \frac{\Gamma(s+\lambda)^3\Gamma(-s)}{\Gamma(s+2\lambda)\Gamma(s+\lambda+\frac{1}{2})} \left(\frac{4z}{(1-z)^2}\right)^s \mathrm{d}\, s}_{\mathcal{O}((1-z)^2(\log(1-z))^2)}.
\end{aligned}
$$

Last contour dented to the right on real axis.

Thus

$$\mathcal{J}^{(\lambda;\lambda-\frac{1}{2})}(z) = \frac{\Gamma(\lambda+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\lambda)} \left(\left(\log\frac{1}{1-z}\right)^2 - 2\left(\gamma - 2\log 2 + \psi(\lambda)\right)\log\frac{1}{1-z} + C \right) + \mathcal{O}\left((1-z)\left(\log(1-z)\right)^2\right).$$

Singularity analysis yields desired result.

- Incomplete Integrals $\int_{-1}^{1-\epsilon} \cdots \mathrm{d}\, t$;
- Jacobi instead of Gegenbauer polynomials;
- Other types of weight functions;

Questions?