Theta functions, ionic crystals energies and optimal lattices

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Talk based on joint works with
M. Faulhuber (University of Vienna) and **H. Knüpfer** (University of Heidelberg)

Point Distributions Webinar
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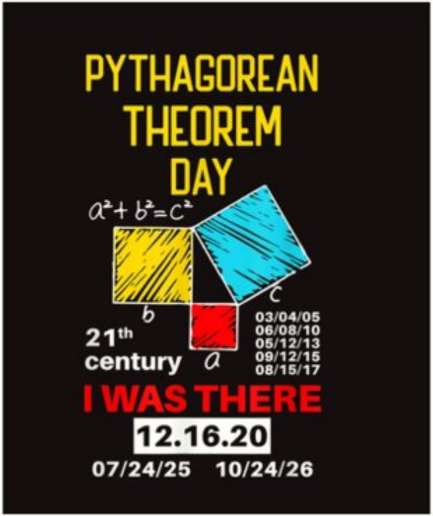
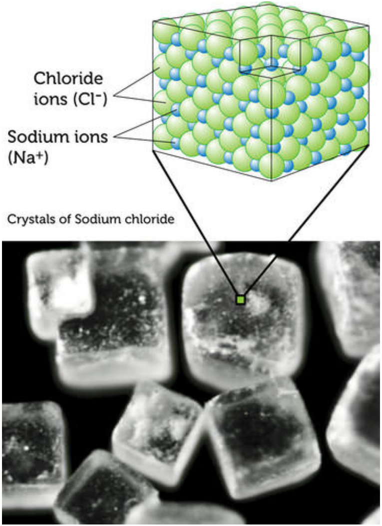
Before starting...



Motivation from (classical) Physics

- Existence of ionic crystals
- Classical solids (not quantum)
- NaCl structure as ground state
- Pairwise interaction potentials
- Minimal energy approach
- Connection potential $\leftrightarrow$ minimizer



Mathematics of crystallization for pairwise potentials

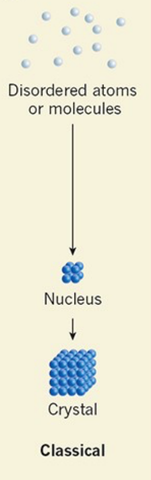
- **Goal:** Determine the minimizer of the potential energy

$$\boldsymbol{E_f(X_N) := \sum_{i\neq j} f(|x_i - x_j|)} \textit{ (possibly in average as } N \to \infty)$$

- **Lennard-Jones potential** $f(r) = r^{-12} - r^{-6}$
  $\Rightarrow$ triangular lattice ground state expected (open problem).

- **Few existing results** for identical particles when $d \geq 2$:
  - Triangular lattice: [Heitmann-Radin '80], (...), [Theil '06]
  - Square lattice: [B.-De Luca-Petrache '19]

- **Wigner conjecture** for Jellium (renormalized Coulomb energy)
  - $d = 2$: crystallization on a triangular lattice (GL vortices);
  - $d = 3$: crystallization on a BCC lattice.
  - ($d \in \{8, 24\}$: proved [Cohn et al. / Petrache-Serfaty '19])



Lattice optimality of multi-component systems

Few results. Only in dimensions $d \in \{1, 2\}$!

- **Short-range potentials 2d w.r.t. charges**:
  - [Radin '86] (quasi-periodicity),
  - [Friedrich-Kreutz '19/'20] (honeycomb and square lattices with alternating charges).
- **2d-torus with graph distance**: [Bouman-Draisma-Van Leeuwaarden '13] (square lattice with alternating charges).
- **1d periodic structures/alternation of charges/3 potentials**: [B.-Knüpfer-Nolte '20].
- **Optimality of lattices with alternation of charges**: [Luo-Wei '20] (3-block copolymers, 2-components BEC)
- **Minimization among lattices with prescribed charges**: [B. '20]
- **Numerical investigations**: Many, especially in 2d.

For more references, visit the

Crystallization Page

https://sites.google.com/site/homepagelaurentbetermin/crystallization-page

Lattices, Interaction Potentials and Charges

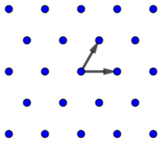
Lattices and two-dimensional examples

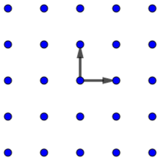
We call a lattice $L \subset \mathbb{R}^d$ any discrete set of the form

$$L = \bigoplus_{i=1}^{d} \mathbb{Z} u_i, \quad \{u_1, ... u_d\} \text{ basis of } \mathbb{R}^d.$$

The set of all these lattices is $\mathcal{L}_d$, and $\mathcal{L}_d(V)$ if the volume of the unit cell is fixed to be $V > 0$. The dual of L is $L^* := \{y \in \mathbb{R}^d : y \cdot p \in \mathbb{Z}, \forall p \in L\}$.

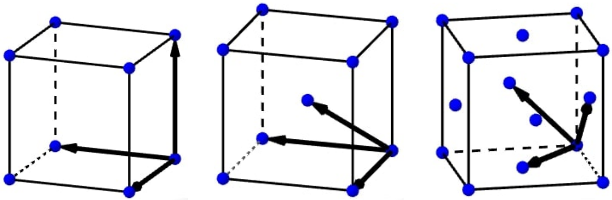
- Square lattice $\mathbb{Z}^2$;
- Triangular lattice $\mathsf{A}_2 = \lambda_0 \left[\mathbb{Z}(1,0) \oplus \mathbb{Z}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right]$, λ_0 s.t. $\mathsf{A}_2 \in \mathcal{L}_2(1)$.





Lattices in dimension $d \geq 3$

- Orthorhombic lattices: generated by an orthogonal basis (e.g. $\mathbb{Z}^d$);
- Body-centred-cubic (BCC) and Face-centred-cubic (FCC) lattices;
- E_8 and the Leech lattice Λ_{24}: best packings [Viazovska et al. '16].



Orthorhombic, BCC and FCC lattices

Remark: Orthorh*=Orthorh, Triang*=Triang, BCC*=FCC, $E_8^* = E_8$, $\Lambda_{24}^* = \Lambda_{24}$.

Pairwise interacting potentials

For technical reasons, we sum the energies on the square of distances.

- Laplace transforms of measures, i.e.

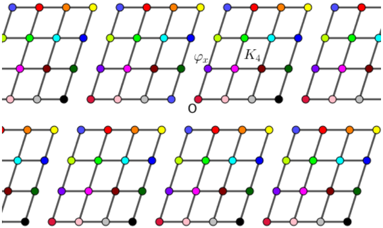
$$f(r) = \mathcal{L}[\mu_f](r) = \int_0^\infty e^{-rt} d\mu_f(t).$$

- Particular case: f is completely monotone if, equivalently:
  - For all $k \in \mathbb{N}_0$, for all $r > 0$, $(-1)^k f^{(k)}(r) \geq 0$;
  - f is the Laplace transform of a positive measure μ_f;
- Typical example: $f(r) = \frac{1}{r^s}$, where $d\mu_f(t) = \frac{t^{s-1}}{\Gamma(s)} dt$.
- Strategy: superposition of Gaussians.
  - $\displaystyle\sum_{p \in L\backslash\{0\}} f(|p|^2) = \int_0^\infty \underbrace{\left(\sum_{p \in L\backslash\{0\}} e^{-|p|^2 t} \right)}_{\text{``theta function''}} d\mu_f(t)$ for suitable f.

Periodic distribution of charges on a lattice L

- **Distribution of charge** $\varphi : L \to \mathbb{R}$, s.t. $x \in L$ has charge $\varphi_x = \varphi(x)$.
- ***N*-periodicity**: $\varphi \in \Lambda_N(L)$, i.e. $\forall x \in L, \forall i$, $\varphi(x + Nu_i) = \varphi(x)$.
- **The total charge is fixed**: $\displaystyle\sum_{y \in K_N} \varphi_y^2 = N^d$. We assume $\varphi_0 > 0$.
- **Periodicity cube** $K_N := \left\{ x = \displaystyle\sum_{i=1}^{d} m_i u_i \in L; 0 \leq m_i \leq N-1 \right\}$.

  We note K_N^* the same for the dual lattice L^*.



Minkowski reduced basis for center and alternating charges

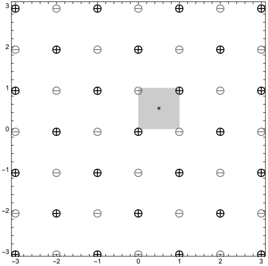
We choose a reduced basis $\{u_1, ..., u_d\}$ for all our L in the following sense:

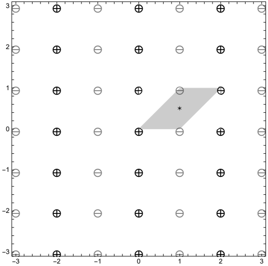
For all $i \in \{1, ..., d\}$, vector u_i satisfies $|u_i| \leq |u_i'|$ for any lattice vector u_i' such that $\{u_1, ..., u_{i-1}, u_i'\}$ can be completed to another basis of L.

$\Rightarrow$ The Minkowski basis of an orthorhombic lattice is the orthogonal one;

$\Rightarrow$ In dimension 2, $|u_1| \leq |u_2|$ and $(\widehat{u_1, u_2}) \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$;

$\Rightarrow$ Good definition of alternating charges distribution and center of the unit cell.





Minkowski basis of $\mathbb{Z}^2$ *vs. another one, with alternation of charges* $\varphi_\pm \in \{\pm 1\}$.

Theta functions associated to lattices

Translated lattice theta function

For a lattice $L \subset \mathbb{R}^d$, a point $z \in \mathbb{R}^d$ and $\alpha > 0$, we define

$$\theta_{L+z}(\alpha) := \sum_{p \in L} e^{-\pi\alpha|p+z|^2}.$$

We have $\theta_{L+z}(\alpha) = \frac{1}{\alpha^{\frac{d}{2}}} P_L\left(z, \frac{1}{4\pi\alpha}\right)$ where P_L solves the **heat equation**

$$\begin{cases} \partial_t u(z,t) = \Delta_z u & \text{for } (z,t) \in \mathbb{R}^d \times (0,\infty) \\ u(z,0) = \sum_{p \in L} \delta_p & \text{for } z \in \mathbb{R}^d. \end{cases}$$

Poisson Summation Formula for Gaussians: $\forall \alpha > 0, L \in \mathcal{L}_d(1), z \in \mathbb{R}^d$,

$$\sum_{p \in L} e^{-\pi\alpha|p+z|^2} = \frac{1}{\alpha^{\frac{d}{2}}} \sum_{p \in L^*} \underbrace{e^{2i\pi p \cdot z}}_{\text{“charge”}} e^{-\frac{\pi|p|^2}{\alpha}}.$$

Two types of optimality for the lattice theta functions

$$L \mapsto \theta_L(\alpha) = \sum_{p \in L} e^{-\pi\alpha|p|^2} \qquad z \mapsto \theta_{L+z}(\alpha) = \sum_{p \in L} e^{-\pi\alpha|p+z|^2}$$

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1. **Optimality among lattices L when $z = 0$.**
   - At fixed density 1 - with respect to $\alpha > 0$.
   - $d = 1$: [Ventevogel '78], minimality $\forall\alpha > 0$ of $\mathbb{Z}$.
   - $d = 2$: [Montgomery '88], minimality $\forall\alpha > 0$ of $\mathbf{A_2}$.
   - $d = 3$: [Sarnak-Strombergsson '06], Conjectured only, **BCC** if $\alpha \leq 1$, **FCC** if $\alpha \geq 1$.
   - $d \in \{8, 24\}$: [Cohn-Kumar-Miller-Radchenko-Viazovska '19], $\mathbf{E_8}$ and the Leech Lattice $\mathbf{\Lambda_{24}}$ $\forall\alpha > 0$ (among periodic configurations).

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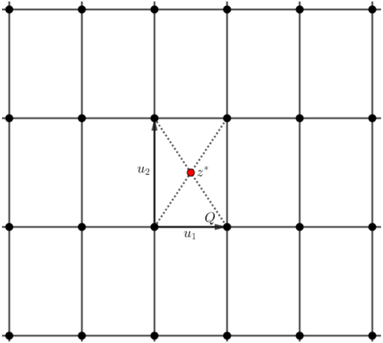
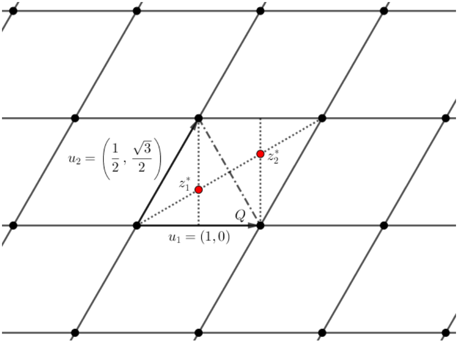
2. **Optimality among vectors z when L is fixed (up to periodicity.)**.
   - $d = 1$: $L = \mathbb{Z}$ - Minimum at the center of an interval $z_0 = 1/2$.
   - Orthorhombic lattices: [B.-Petrache '17] - Center of the cell z_0.
   - Triangular lattice: [Baernstein II '97] - Barycenter of a triangle z_0.

Optimality among vectors z when L is fixed

$$\text{Minimizer of} \quad z \mapsto \theta_{L+z}(\alpha) = \sum_{p \in L} e^{-\pi\alpha|p+z|^2} \quad \forall \alpha > 0.$$





Proof of Born's Conjecture

L. Bétermin and H. Knüpfer, **On Born's conjecture about optimal distribution of charges for an infinite ionic crystal**, *Journal of Nonlinear Science* 28, 2018.

Born's problem for the electrostatic energy (1921)

Max Born (1882-1970)

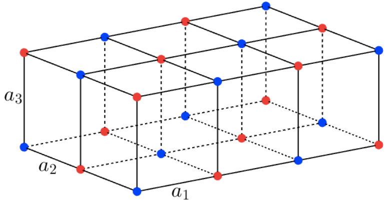
"How to arrange positive and negative charges
on a simple cubic lattice
so that the electrostatic energy is minimal?"

Über elektrostatische Gitterpotentiale,
Zeitschrift für Physik, 7:124-140, 1921

Born's Conjecture (1921) for Coulomb potential

Conjecture [Born '21]

The alternate distribution of charges $\varphi_\pm = \pm 1$ (NaCl) is the unique solution among all periodic distributions of charges $\varphi \in \bigcup_N \Lambda_N(\mathbb{Z}^3)$.



The total amount of charge is fixed and the **neutrality** has to be assumed (the Coulomb potential is not summable on $\mathbb{Z}^3$).

Born's proofs: 1d and local minimality in 3d for Coulomb interaction.

Energy in terms of translated theta function

If $r \mapsto f(r^2) \in \ell^1(L\backslash\{0\})$ and $\mu_f \geq 0$, then, using the Discrete Fourier Transform,

$$\mathcal{E}_{L,f}[\varphi] = \lim_{\eta \to 0} \left(\frac{1}{2N^d} \sum_{p \in L\backslash\{0\}} \sum_{y \in K_N} \varphi_y \varphi_{p+y} f(|p|^2) e^{-\eta |p|^2} \right)$$
$$= \frac{1}{2N^d} \sum_{k \in K_N^*} \xi_k E[k], \quad \xi_k := N^{-\frac{d}{2}} \widecheck{\varphi * \varphi}(k) \geq 0$$

where, using $f(r) = \mathcal{L}[\mu_f](r)$ and the Poisson Summation Formula,

$$E[k] = \sum_{p \in L\backslash\{0\}} e^{-\frac{2\pi i}{N} p \cdot k} f(|p|^2) = \int_0^\infty \left(\pi^{\frac{d}{2}} t^{-\frac{d}{2}} \theta_{L^* + \frac{k}{N}}\left(\frac{\pi}{t}\right) - 1 \right) d\mu_f(t).$$

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Strategy (general): Finding the minimizer z_0 of $z \mapsto \theta_{L^*+z}(\alpha)$ for all $\alpha > 0$ and such that $\exists N_0 > 0$ where $z_0 \in N_0^{-1} L^*$ gives the solution (ξ and then φ).

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Remark: *Also true for non-summable $f(r^2)$ if $\sum_{y \in K_N} \varphi_y = 0$ (Ewald Summation Method).*

Proof of Born's Conjecture

Our general result works for any completely monotone potential f.

- If L^* is orthorhombic, then $z_0 = \frac{1}{2}\sum_{i=1}^d u_i^*$ is the center and $N_0 \in 2\mathbb{N}$.
- If $L^* = \mathsf{A}_2^* = \mathsf{A}_2$, then z_0 is a barycenter of a triangle and $N_0 \in 3\mathbb{N}$.

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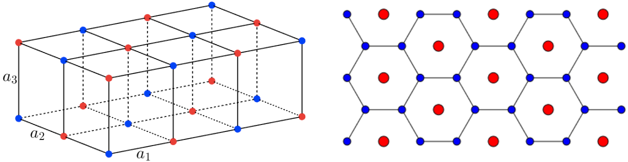
Theorem - Solution of Born's problem [B.-Knüpfer '17]

The unique minimizer of $\varphi \mapsto \mathcal{E}_{L,f}[\varphi]$ in $\bigcup_N \Lambda_N(L)$ is:

- The alternate distribution $\varphi_\pm$ of charges -1 and $+1$ if L is orthorhombic;
- The honeycomb-like distributions of charges $\left\{-\frac{\sqrt{2}}{2}, +\sqrt{2}\right\}$ if $L = \mathsf{A}_2$.



Rock-Salt ($NaCl$) structure and Sodium Sulfide (Na_2S) layer

Maximal Theta Function among alternated lattices

L. Bétermin and M. Faulhuber, **Maximal Theta Functions - Universal Optimality of the Hexagonal Lattice for Madelung-Like Lattice Energies**, arXiv:2004.04553, 2020.

Alternate and centred lattice theta functions

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- There is no minimizer for $L \mapsto \mathcal{E}_{L,f}[\varphi_\pm]$ if $\mu_f \geq 0$.
- **Problem:** Finding the maximizer in $\mathcal{L}_d(1)$, for fixed $\alpha > 0$, of the alternate lattice theta function,

$$L \mapsto \theta_L^\pm(\alpha) := \sum_{p\in L} \varphi_\pm(p) e^{-\pi\alpha|p|^2}.$$

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- Equivalently, by Poisson Summation Formula, we want to maximize in $\mathcal{L}_d(1)$, for fixed $\alpha > 0$, the centred lattice theta function

$$L \mapsto \theta_L^c(\alpha) = \sum_{p\in L} e^{-\pi\alpha|p+c_L|^2}, \quad c_L := \frac{1}{2}\sum_{i=1}^d u_i.$$

A new universal optimality result

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Theorem - Maximality of the triangular lattice [B.-Faulhuber '20]

For any $\alpha > 0$, A_2 is the unique maximizer of $\theta_L^{\pm}(\alpha)$ and $\theta_L^{c}(\alpha)$ in $\mathcal{L}_2(1)$.

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- It is a new type of universal optimality among lattices, i.e. true for any completely monotone potential (see [Cohn-Kumar '07]).

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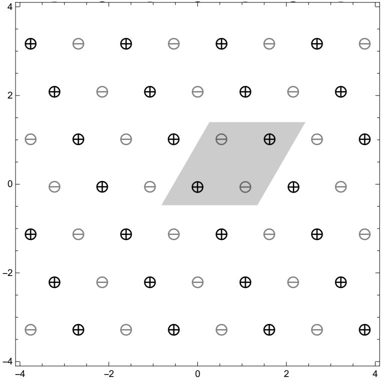
For any $\alpha > 0$, A_2 is the unique maximizer of $\theta_L^{\pm}(\alpha)$ and $\theta_L^c(\alpha)$ in $\mathcal{L}_2(1)$.

- Direct consequence: the same is true for all f s.t. $\mu_f \geq 0$, i.e. A_2 maximizes

$$L \mapsto \mathcal{E}_{L,f}[\varphi_{\pm}] = \sum_{p\in L\backslash\{0\}} \varphi_{\pm}(p) f(|p|^2) = \int_0^\infty \sum_{p\in L\backslash\{0\}} \varphi_{\pm}(p) e^{-|p|^2 t} d\mu_f(t).$$

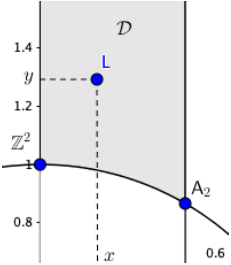
- It is a new type of universal optimality among lattices, i.e. true for any completely monotone potential (see [Cohn-Kumar '07]).
- Conjectured to be true in dimensions $d \in \{8, 24\}$ for E_8 and Λ_{24}.

The maximal structure $(\mathsf{A}_2, \varphi_\pm)$ amongst $(L, \varphi_\pm)$



Maximality of A_2 - Idea of the proof

1. We parametrize L by a point (x, y) in the half-elliptic domain
$$\mathcal{D} := \{(x, y) \in \mathbb{R}^2 : x \in [0, 1/2], y > 0, x^2 + y^2 \geq 1\}.$$
2. We show that $x \mapsto \theta_L^{\pm}(\alpha)$ is increasing on $[0, 1/2]$;
3. Fixing $x = 1/2$, we show that $y \mapsto \theta_L^{\pm}(\alpha)$ is decreasing on $\left[\frac{\sqrt{3}}{2}, \infty\right)$.
4. Use of estimates comparable to the one of [Montgomery '88].



Optimality of the rock-salt structure among lattices and charges

L. Bétermin, M. Faulhuber and H. Knüpfer, **On the optimality of the rock-salt structure among lattices with charge distributions**, accepted in *Mathematical Models and Methods in Applied Sciences*, arXiv:2004.04553, 2020.

A new energy E_{f_1,f_2} with repulsion at 0

- We remark that, if $\mu_f \geq 0$, then $\min_{L,\varphi} \mathcal{E}_{L,f}[\varphi] = -\infty$.
- We add a completely monotone repulsive part f_1, defining a new f:

$$f(|x-y|^2) = \underbrace{f_1(|x-y|^2)}_{\text{Repulsion at the origin}} + \underbrace{\varphi_x \varphi_y f_2(|x-y|^2)}_{\text{Tail - Charges interaction}}.$$

- Chosen potentials: $f_1(r) = r^{-p}$, $f_2(r) = r^{-q}$, such that $p > q > d/2$.
- New total energy per point:

$$E_{f_1,f_2}[L,\varphi] := \underbrace{\sum_{p \in L\backslash\{0\}} f_1(|p|^2)}_{E_{f_1}[L]} + \underbrace{\frac{1}{N^d} \sum_{p \in L\backslash\{0\}} \sum_{y \in K_N} \varphi_y \varphi_{p+y} f_2(|p|^2)}_{2\mathcal{E}_{L,f_2}[\varphi]}.$$

- **Problem:** Find the minimizer of E_{f_1,f_2} in $\mathcal{L}_d \times \bigcup_N \Lambda_N(\mathcal{L}_d)$.
  Is it a rock-salt structure $(\lambda\mathbb{Z}^d, \pm 1)$? (for some $\lambda > 0$)

The two competitive energies

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- If $\mu_{f_1} \geq 0$, E_{f_1} has a **minimizer** in $\mathcal{L}_d(V)$ for fixed $V > 0$ (not in $\mathcal{L}_d$):
  - $d = 2$: A_2 for all $V > 0$.
  - $d = 3$: BCC or FCC (conjecture), probably changing with V.
  - $d \in \{8, 24\}$: E_8 and the Leech lattice Λ_{24}.
  - **Among orthorhombic lattices:** $V^{\frac{1}{d}}\mathbb{Z}^d$ is the unique minimizer in $\mathcal{L}_d(V)$, [Montgomery '88].

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- If $\mu_{f_2}\geq 0$, $(L,\varphi)\mapsto\mathcal{E}_{L,f_2}[\varphi]$ does not have any minimizer, only **maximizers**.
  - If L is fixed, possibility to find the optimal φ (Born's problem).
  - If $\varphi=1$, then $\mathcal{E}_{L,f_2}[\varphi]=E_{f_2}[L]$ (see above), but not optimal.
  - Among all lattices of $\mathcal{L}_2(V)$ with $\varphi_\pm$, A_2 is the unique maximizer.
  - **Among orthorhombic lattices with $\varphi_\pm$**, $V^{\frac{1}{d}}\mathbb{Z}^d$ is the unique maximizer in $\mathcal{L}_d(V)$ [Faulhuber-Steinerberger '17].

Orthorhombic lattices with $\varphi_\pm$ at fixed density

$$E^{\pm}_{f_1,f_2}[L] := E_{f_1,f_2}[L,\varphi_\pm] = \sum_{p\in L\backslash\{0\}} f_1(|p|^2) + \sum_{p\in L\backslash\{0\}} \varphi_\pm(p) f_2(|p|^2)$$

Born: Restricted to orthorhombic lattices, $E^{\pm}_{f_1,f_2}[L] \leq E_{f_1,f_2}[L,\varphi]\ \forall N, \forall \varphi \in \Lambda_N(L)$.

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Theorem - Optimality of the rock-salt structure [B.-Faulhuber-Knüpfer '20]

Let $f_1(r) = r^{-p}$ and $f_2(r) = r^{-q}$, $p > q > d/2$.

- $\forall d \geq 1$, $V^{\frac{1}{d}}\mathbb{Z}^d$ is a critical point of $E^{\pm}_{f_1,f_2}$ in $\mathcal{L}_d(V)$.
- $\forall d \geq 1$, there exists V_0 such that for all $V \in (0, V_0)$, $V^{\frac{1}{d}}\mathbb{Z}^d$ is the **unique minimizer** of $E^{\pm}_{f_1,f_2}$ among orthorhombic lattices with fixed volume $V > 0$.
- $\forall d \geq 1$, there exists V_1 such that for all $V \in (V_1, \infty)$, $V^{\frac{1}{d}}\mathbb{Z}^d$ is **not a minimizer** of $E^{\pm}_{f_1,f_2}$ among orthorhombic lattices with fixed volume $V > 0$.

In particular, restricted to orthorhombic lattices, the rock-salt structure is globally optimal among charges and lattices at high density.

Global optimality of the rock-salt structure

We define the alternate Epstein zeta function by

$$\zeta_L^{\pm}(s) := \sum_{p\in L\backslash\{0\}} \frac{\varphi_{\pm}(p)}{|p|^s}, \quad s > d.$$

We already know that $\zeta_L^{\pm}(s) < \zeta_{\mathsf{A}_2}^{\pm}(s) < 0$ for all $L \in \mathcal{L}_2(1)$ by [B.-Faulhuber '20]. This is expected to be true for all $d \geq 1$.

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Theorem - Minimal energy among dilated [B.-Faulhuber-Knüpfer '20]

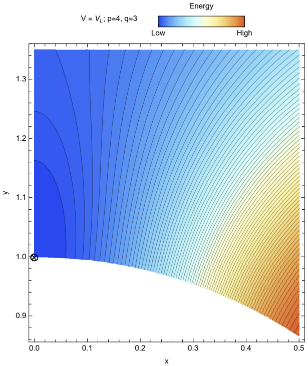
Let $d \geq 1$, $f_1(r) = r^{-p}$ and $f_2(r) = r^{-q}$, $p > q > d/2$. For $L \in \mathcal{L}_d(1)$, if $\zeta_L^{\pm}(2q) < 0$, then

$$\mathcal{E}_L^{\pm} := \min_{V>0} E^{\pm}_{f_1,f_2}[V^{\frac{1}{d}}L] = \frac{(q-p)\left(-q\zeta_L^{\pm}(2q)\right)^{\frac{p}{p-q}}}{qp(p\zeta_L(2p))^{\frac{q}{p-q}}} < 0.$$

Problem: What is the minimizer of $L \mapsto \mathcal{E}_L^{\pm}$ in $\mathcal{L}_d(1)$? (shape of the global min)

Numerics in dimension 2 for $f_1(r^2) = r^{-8}$, $f_2(r^2) = r^{-6}$.

We plot $L \mapsto \mathcal{E}_L^{\pm}$ in the fundamental domain $\mathcal{D}$ of 2d lattices.



Numerically, the global minimum of $E^{\pm}_{f_1,f_2}$ in $\mathcal{L}_2$ is a square lattice.

The same is expected in dimensions $d \in \{3, 8, 24\}$ for $\mathbb{Z}^3$, $\mathbb{Z}^8$ and $\mathbb{Z}^{24}$. We have compared different values of $\mathcal{E}_L^{\pm}$.

Other numerical observations

- We have compared $\mathcal{E}^{\pm}_{\mathbb{Z}^d}$ with other optimal charge distributions, without finding any better candidate for being a minimizer of $(L,\varphi) \mapsto E_{f_1,f_2}[L,\varphi]$.

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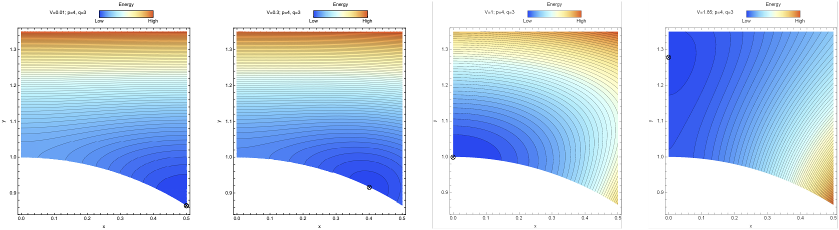
- As V increases, $\underset{L \in \mathcal{L}_2(V)}{\operatorname{argmin}} E^{\pm}_{f_1,f_2}$ exhibits a phase transition of type:

Triangular - Rhombic - Square - Rectangular.



Already observed for Lennard-Jones/Morse potentials, 3-block copolymers, two-components Bose-Einstein-Condensates.

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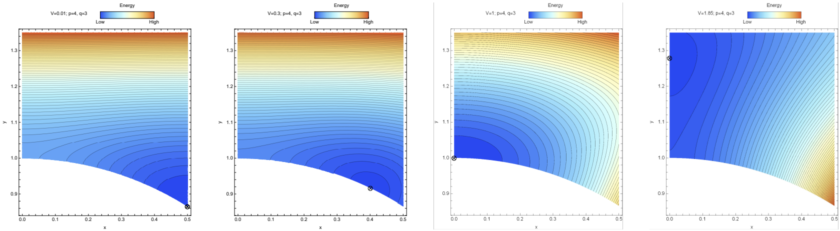
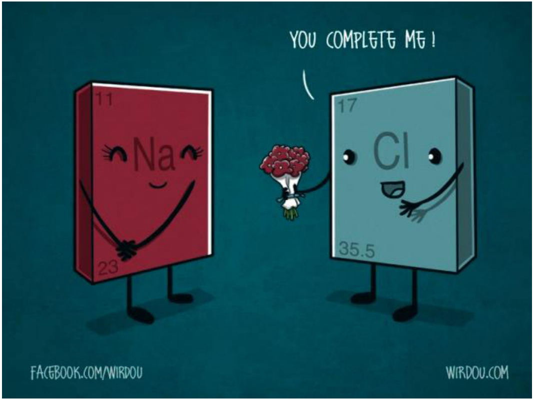


  Already observed for Lennard-Jones/Morse potentials, 3-block copolymers, two-components Bose-Einstein-Condensates.

- If $p - q$ is small enough, the minimizer of $E^{\pm}_{f_1,f_2}$ in $\mathcal{L}_2$ is not a square lattice.

Conclusion

- New general strategy to optimize energies among charges on a lattice.
  - Key point: minimization of $z \mapsto \theta_{L^*+z}(\alpha)$ for all $\alpha > 0$.
- New universal optimality property for the triangular lattice.
  - Key point: minimization of $L \mapsto \theta_L^{\pm}(\alpha)$ for all $\alpha > 0$.
- New type of minimization problem for $\mathbb{Z}^d$ - NaCl ground state.
  - Similar to minimizing $L \mapsto \theta_L(\beta) + \theta_L^{\pm}(\alpha)$, $\beta > \alpha$.



Thank you for your attention!