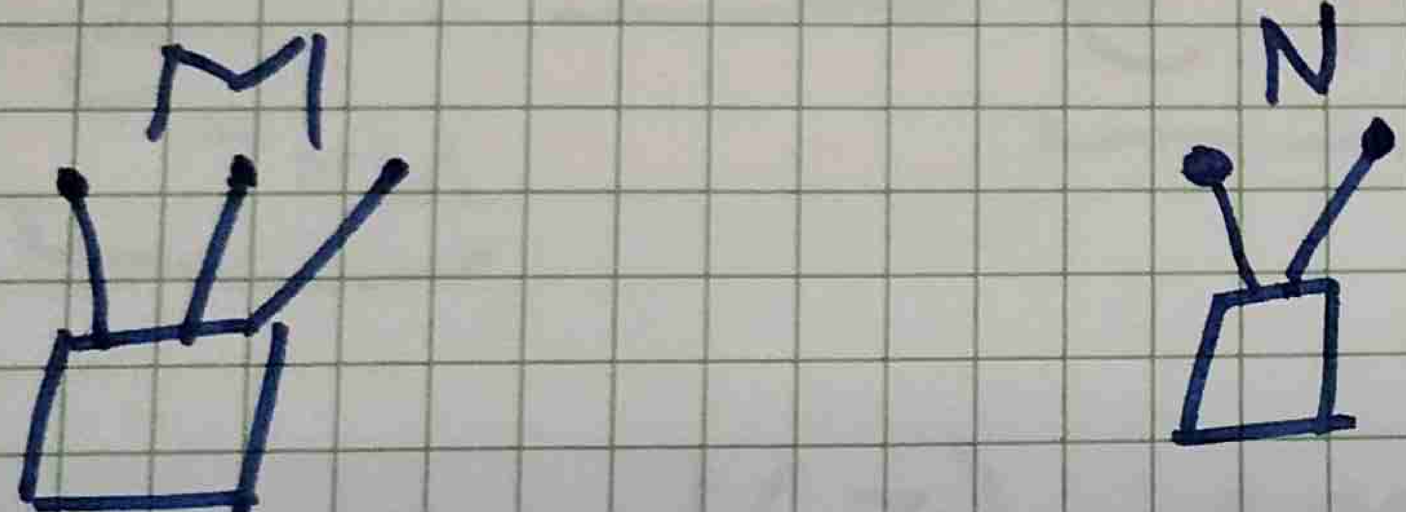
Distributing many points in $\mathbb{G}(M, \mathbb{C}^T)$ & application to communic.

Carlos Beltrán
Universidad de Cantabria

Joint: Diego Cuevas, Javier Álvarez Vizoso, Nacho Santamaría
Vic Tuček, Gunnar Peters




$$\begin{array}{l} 1\ \text{time} \to \\ \vdots \\ T\ \text{time} \end{array} \left( \begin{matrix} z_1^1 & \cdots & z_M^1 \\ \vdots & & \vdots \\ z_1^T & \cdots & z_M^T \end{matrix} \right) \curvearrowright (z_1 \dots z_M) \cdot \overbrace{\left( H \right)}^{N} \Big]\, M + \sqrt{\frac{M}{\rho}}\, W$$

$H$ ↑ Doppler effect

$W \to N(0,1)$

Bands

$\boxed{T = 1000}$

$\boxed{T = 10}$

$= Z \longrightarrow \underset{T \times M}{Z} \cdot \underset{M \times N}{H} + \sqrt{\frac{M}{T\rho}}\, W$

$\boxed{M = N}$ Received $Z \cdot H + \text{Noise}$. If Noise $<< 1$ $\approx Z \cdot H$

Coherent: Agree: $Z = \begin{pmatrix} I_M \\ * \end{pmatrix}$ receive $= \begin{pmatrix} * \\ * \end{pmatrix}$ $\begin{pmatrix} I_M \\ * \end{pmatrix} H \quad H^{-1}$

$T = 80$

$M = N = 2$

$$\underbrace{Z}_{T \times M} \longrightarrow \underbrace{Z \cdot \overset{?}{H}} + \underset{\substack{\uparrow \\ 1}}{\text{Noise}} \longrightarrow \text{Recognizes class of } Z \text{ in } G(M, \mathbb{C}^T)$$

span-col$(Z)$ = span-cols$(ZH)$

$G(M, \mathbb{C}^T)$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} \to X_1, \ldots, X_N \qquad \xrightarrow{X_1} \qquad X_1 \cdot H + W = Y \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Find $i$ that max $\operatorname{tr}(Y^H X_i X_i^H Y)$

Works better if the $X_i$ are well sep.

Choose $X_1, \dots, X_N$ that maximize

$$\min_{i \neq j} d_{chordal}(X_i, X_j)$$

Jasper, King, Mixon '19

Game of Sloanes

Riemannian optimization.

$$f: \mathbb{P}(\mathbb{C}^T) \to \mathbb{R}$$

$$\dot{z} \to \left(\overline{z + t\dot{z}}\right)$$

choose $(\|z\| = 1)$

$$z \in \mathbb{P}(\mathbb{C}^T). \quad T_z \mathbb{P}(\mathbb{C}^T) \equiv z^{\perp} \subseteq \mathbb{C}^T$$

$$T_z \mathbb{P}(\mathbb{C}^T) = \left\{ \gamma : (-1, 1) \to \mathbb{P}(\mathbb{C}^T), \ \gamma(0) = z \right\} / \sim$$

$\gamma_1 \sim \gamma_2$ if $\forall g : \mathbb{P}(\mathbb{C}^T) \to \mathbb{R}$,

$$\frac{d}{dt}\Big|_{t=0} g(\gamma_1(t)) = \frac{d}{dt}\Big|_{t=0} g(\gamma_2(t))$$

$$\hat{f} : \mathbb{C}^T \longrightarrow \mathbb{R}, \quad \hat{f}(z) = \hat{f}(\lambda z)$$

$$f : \mathbb{P}(\mathbb{C}^T) \longrightarrow \mathbb{R} \qquad f(z) = \hat{f}\left(\frac{z}{\|z\|}\right)$$

$$\cancel{\mathrm{Re}}\langle \underbrace{\nabla f(z)}_{\in z^\perp}, \underbrace{\dot{z}}_{\in z^\perp} \rangle = Df(z)\dot{z}$$

$$Df(z)\dot{z} = \frac{d}{dt}\Big|_{t=0} f(z + t\dot{z}) = \frac{d}{dt}\Big|_{t=0} \hat{f}\left(\frac{z+t\dot{z}}{\|z+t\dot{z}\|}\right) = D\hat{f}(z) \cdot \underbrace{\frac{d}{dt}\Big|_{t=0} \frac{z+t\dot{z}}{\|z+t\dot{z}\|}}_{\dot{z}} = D\hat{f}(z)\dot{z}$$

$\dot{z} \perp z$

$\|z\| = 1$

$$\cancel{\mathrm{Re}}\langle \underline{D\hat{f}(z)}, \dot{z} \rangle = D\hat{f}(z)\dot{z}$$

$$\boxed{\Pi_{z^\perp}(\nabla f(z))}$$

$$d_{ch}(z, w) = \sqrt{1 - |z^* w|^2} \qquad \mathbb{P}(\mathbb{C}^T) \longrightarrow \mathbb{C}^{T \times T}$$
$$z \rightsquigarrow z z^*$$

$$\hat{f} : \mathbb{C}^T \longrightarrow \mathbb{R}$$
$$z \rightsquigarrow \sqrt{1 - |z_1|^2} \qquad \hat{f}(z) = \hat{f}(\lambda z), \ \forall |\lambda| = 1.$$

$$f : \mathbb{P}(\mathbb{C}^T) \longrightarrow \mathbb{R}$$
$$\underline{z} \rightsquigarrow d_{chordal}\left(z, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \sqrt{1 - |z_1|^2}$$

$$\mathrm{Re} \langle \nabla \hat{f}(z), \underset{\mathbb{C}^T}{\underset{\cap}{\dot{z}}} \rangle = D\hat{f}(z)\dot{z} = \frac{d}{dt}\Big|_{t=0} \hat{f}(z + t\dot{z})$$

$$= \frac{d}{dt}\Big|_{t=0} \sqrt{1 - |z_1 + t\dot{z}_1|^2}$$

$$\nabla \hat{f}(z) = \frac{\begin{pmatrix} z_1 \\ 0 \end{pmatrix}}{d\left(z, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}$$

$$= \frac{\mathrm{Re}(\dot{z}_1, z_1)}{\sqrt{1 - |z_1|^2}} = \mathrm{Re}\left\langle \frac{\begin{pmatrix} z_1 \\ 0 \end{pmatrix}}{d\left(z, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}, z_1 \right\rangle$$

$$\underline{\nabla f(z)} = \frac{\begin{pmatrix} z_1 \\ 0 \end{pmatrix}}{d\left(z, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)} - \frac{\left\langle z, \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \right\rangle z}{d\left(z, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)} \in z^\perp$$

$$z^{(0)}$$

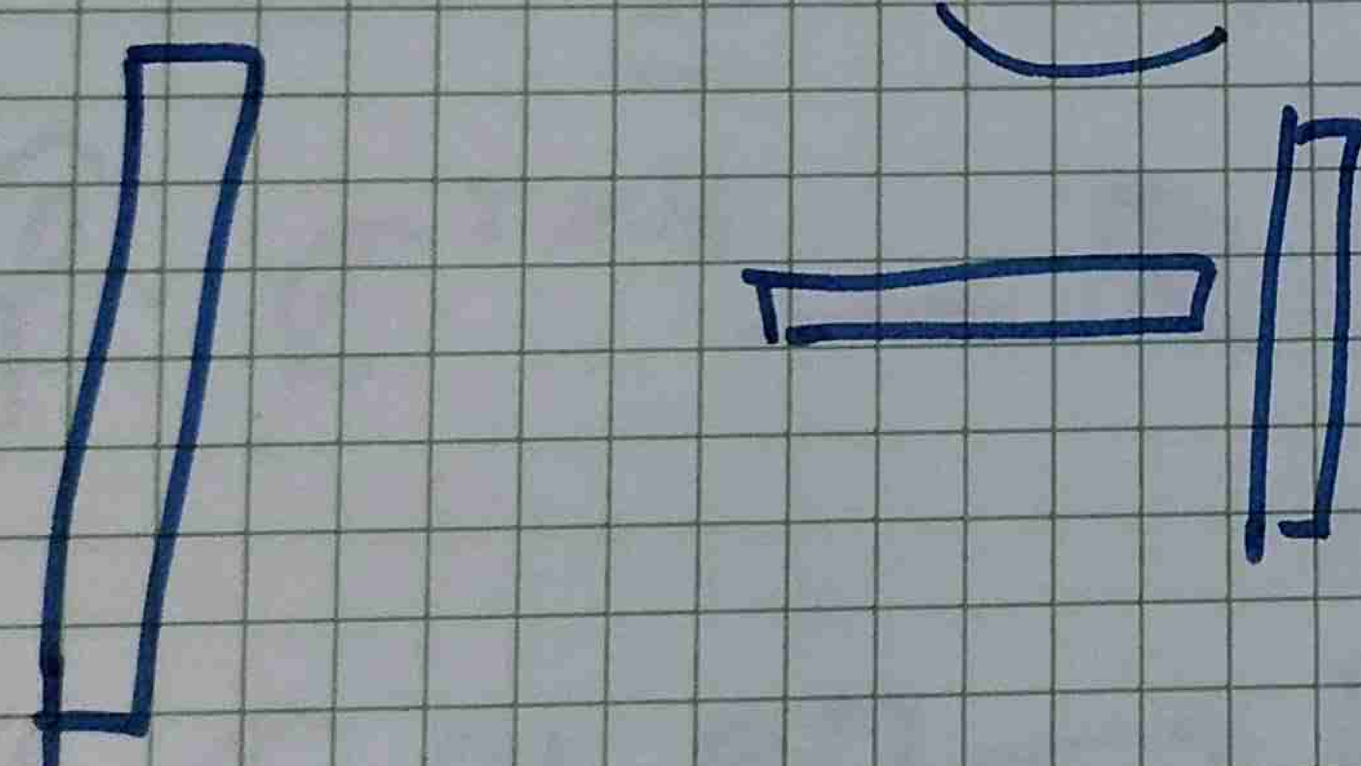
$$z^{(1)} = \underline{z^{(0)}} + \underline{t_k} \cdot \nabla f(z^{(0)}) \longrightarrow z_*^{(1)} = \frac{z^{(1)}}{\| z^{(1)} \|}$$

$\mathcal{G}(M, \mathbb{C}^T)$. Where did "$\mathcal{Z}^\perp$" come from?

$$\Pi : \mathrm{Stiefel}(M, \mathbb{C}^T) \longrightarrow \mathcal{G}(M, \mathbb{C}^T) = \mathrm{Stief} / \mathcal{U}(M\times M)$$

$$X \longmapsto \text{span of cls. of } X.$$

$$\{ X \in \mathbb{C}^{T\times M} : X^* X = I_M \}$$



$$T_X\, \mathcal{G}(M, \mathbb{C}^T) = T_X\, \mathrm{Stiefel}(M, \mathbb{C}^T) \cap \{ \dot{X} \text{ orthogonal to the orbit } \{ X \cdot U : U \text{ unitary} \} \}$$

$$\boxed{\begin{gathered} T_X\, \mathcal{G}(M, \mathbb{C}^T) \\ \equiv \\ \{ \dot{X} \in \mathbb{C}^{T\times M} : X^* \dot{X} = 0 \} \end{gathered}}$$

$\mathbb{G}(M, \mathbb{C}^T)$. dim: $M(T-M)$

vol ✓

$T_X \mathbb{G}(M, \mathbb{C}^T) \equiv \dot{X} \in \mathbb{C}^{T \times M} : \underline{X^* \dot{X} = 0_{M \times M}}$

$\underline{X} + t \cdot \dot{X}$

$\hat{f} : \mathbb{C}^{T \times M} \longrightarrow \mathbb{R}$ $\quad \underline{\hat{f}(X) = \hat{f}(X \cdot U)}$ for unitary $U$.

$\nabla \hat{f}(X)$

$\downarrow$

$\Pi \quad \nabla \hat{f}(X)$

$\forall \dot{X} : X^* \dot{X} = 0_M$

$=$

$\boxed{\underbrace{(I - X X^*)}_{T \times T} \nabla \hat{f}(X)}$

$f : \mathbb{G}(M, \mathbb{C}^T) \longrightarrow \mathbb{R}$

$[X] \longrightarrow \underline{\hat{f}(X \cdot \underline{(X^* X)^{-1/2}})}$

$X = U \begin{pmatrix} D \\ 0 \end{pmatrix} V^* \qquad X^* X = V (D)^2 V^*$

repr in Stiefel $(\underline{X^* X} = I_M)$

$$\underline{f : G(M, \mathbb{C}^T) \longrightarrow \mathbb{R}} \qquad // \ \hat{f} : \mathbb{C}^{T\times M} \longrightarrow \mathbb{R}$$

$\hat{f}(X) = \hat{f}(XU)$, $U$ unitary.

$$\underbrace{X^{(1)}} = X^{(0)} + h \underbrace{\nabla f(X^{(0)})}$$

$$= \left(I - X^{(0)} X^{(0)*}\right) \cdot \nabla \hat{f}(X^{(0)})$$

$$\cancel{\frac{X^{(1)}}{\| X^{(1)} \|}}$$

$$\boxed{X^{(1)} = Q} \cdot \cancel{R}$$

$$\underline{f : G(M, \mathbb{C}^T)^N} \longrightarrow \mathbb{R}$$

$$\begin{matrix} X_1 \\ \vdots \\ X_T \end{matrix} \longrightarrow \left\{ \begin{array}{l} \sum_{i \neq j} d_{ch}(X_i, X_j) \\ \min_{i \neq j} d(X_i, X_j) \\ \_\_\_\_ \end{array} \right.$$

$$\min d(X_i, X_j) \approx \frac{C}{N^{\frac{1}{2M(T-M)}}}$$

High noise, 2 points.

$$P_{\text{error}} \longrightarrow \frac{1}{2} - \frac{T N^{1/2} d_{\text{chordal}}(X_1, X_2)}{4M} \cdot \rho$$

$$\underset{T \times M}{X} = \underset{\substack{\text{unit.} \\ T \times T}}{U} \begin{pmatrix} \overset{\text{diag}}{D} \\ 0 \end{pmatrix} \cdot \underset{\substack{\text{unit.} \\ M \times M}}{V}$$

$$X \Rightarrow \underbrace{X \cdot (X^* X)^{-1/2}}_{"Y}$$

$$Y^* Y = I$$

$$\text{span} = \text{span}(U^1 \dots U^M)$$