Asymptotics of Best Packing and Best Covering

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June 2021

Statement of the problem

- Given a compact set, what is the maximal number of disks which can be packed into the set?

- Similarly, there is a dual problem which asks for the minimal of disks needed to cover a given compact set.

- Examples of these questions, how will a given gas expand to fit its container? How do you best distribute an important resource over a large area with irregular shape?

Examples

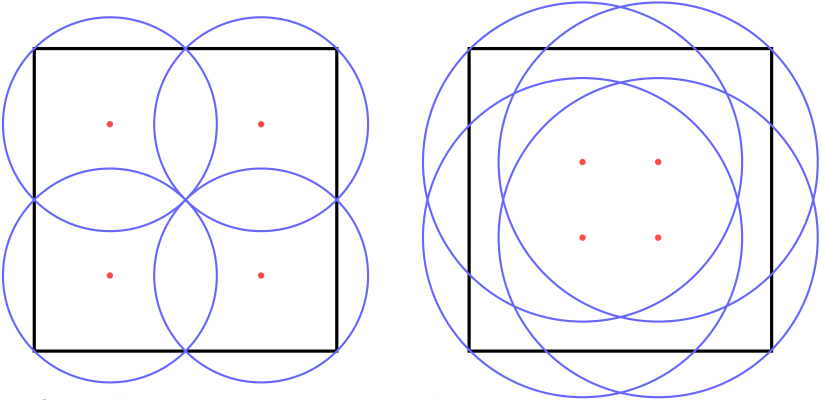


Figure: On the left we have a well covered square with four points, on the right a poorly covered square by four points

Best Packings

- Let $\mathbb{R}^d$ be our ambient space and

$$\omega_N = \{y_1, \ldots, y_N\} \subset \mathbb{R}^d$$

a collection of N points

- Set the following to be the minimum distance between any pair of points

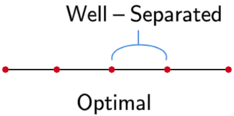
$$\delta(\omega_N) := \min_{1 \leq i \neq j \leq N} |y_i - y_j|$$

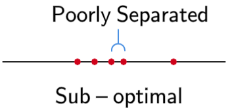
Best Packings

- Let A be an infinite, compact set and define the best packing distance of N points on A as

$$\delta_N(A) := \sup\{\delta(\omega_N) : \omega_N \subset A, \#\omega_N = N\}$$

- For instance the best packing of N points in $[0, 1]$ consists of N equally spaced points with separation $\frac{1}{N-1}$





Best Coverings

- Let A be an infinite, compact set and define the best (constrained) covering distance of N points on A as

$$\rho_N(A) := \min_{\omega_N \subset A} \max_{y \in A} dist(y, \omega_N)$$

- If we consider the collection of points $\omega_N^* \subset \mathbb{R}^d$ then we call this the **unconstrained covering**, denoted $\rho_N^*(A)$

$$\rho_N^*(A) := \min_{\omega_N^* \subset \mathbb{R}^d} \max_{y \in A} dist(y, \omega_N^*)$$

Minimal Energy

- Given a lower semi-continuous kernel $K(x, y) : A \times A \to (-\infty, \infty]$ and a configuration ω_N given as seen previously, its $K-$energy is given as
$$E_K(\omega_N) := \sum_{1 \le i \ne j \le N} K(x_i, x_j)$$
$$\mathcal{E}(A; N) := \min_{\omega_N \subset A} \{E_K(\omega_N)\}$$
the minimal $K-$energy over all such configurations
- Finding the optimal configurations such that $E = \mathcal{E}$ is in general a difficult problem

Maximal Polarization

- The problem of maximal polarization is defined as follows

$$U_K(y; \omega_N) := \sum_{i=1}^{N} K(y, x_i)$$

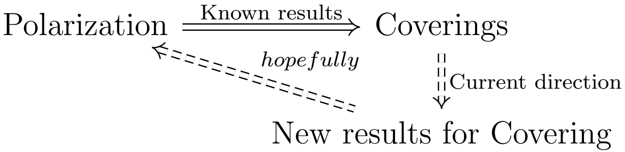
then consider the minimum

$$P_K(A; \omega_N) := \inf_{y \in A} U_K(y; \omega_N)$$

then the Nth $K-$polarization (Chebyshev) constant of A is defined as

$$\mathcal{P}_K(A; N) := \sup_{\omega_N \subset A} P_K(A; \omega_N)$$

What's the Goal?



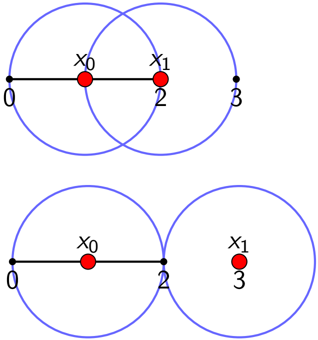
Theorem:

Theorem

For compact sets $A \subset \mathbb{R}^p$ and integers d, p satisfying $1 \leq d \leq p$ and $2 \leq p$ and $\mathcal{H}_d(A) > 0$, then: If $\lim_{N\to\infty} \eta_N^*(A) \cdot N^{1/d}$ *exists, then* $\lim_{N\to\infty} \eta_N(A) \cdot N^{1/d}$ *also exists and they agree.*

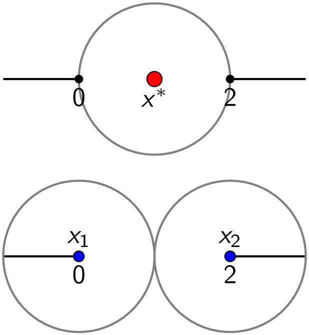
Weak Separation for Coverings

- We need a lemma that gives us flexibility in our covering points



Replacing Uncontrained with Constrained points

- If a point lies outside of the set, we can replace it with nearby points



Consequences and Transition

- This result, and its contrapositive, has additional results worth discussing

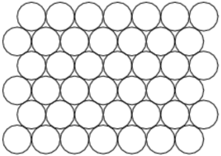
Densest Sphere Packing in $\mathbb{R}^p$

- What is the 'densest' arrangement of congruent p-dimensional spheres in $\mathbb{R}^p$? (by proportion of volume the spheres occupy in the ambient space.) Denote the quantity by Δ_p.
- Prior to 2015 we knew only cases $p = 1, 2, 3$
- Maryna Viazovska, Henry Cohn & collaborators recently showed:

E8 attains $\Delta_8 = \pi^4/324$ in $\mathbb{R}^8$

Leech lattice attains $\Delta_{24} = \pi^8/(12!)$ in $\mathbb{R}^{24}$

Figure: **Hexagonal Lattice** attains $\Delta_2 = \pi/\sqrt{12}$ in $\mathbb{R}^2$

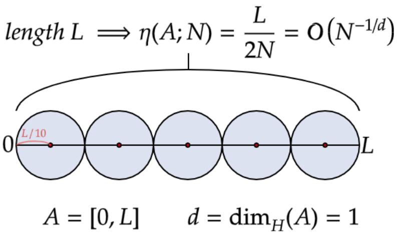


hexagonal packing

Growth Order of Packing and Covering

- Proving *exact* answers is generally very hard $\implies$ consider $N \to \infty$
- Let $d = \dim_H(A)$ denote the **Hausdorff dimension** of $A \subset \mathbb{R}^p$ compact. Then

$$\delta(A; N) = \mathcal{O}(N^{-1/d}), N \to \infty,$$
$$\eta(A; N) = \mathcal{O}(N^{-1/d}), N \to \infty,$$
$$\eta^*(A; N) = \mathcal{O}(N^{-1/d}), N \to \infty.$$



Poppy-Seed Bagel Theorem (PSBT)

- **Smoothness** assumptions on $A \implies$ existence of asymptotics for best packing, best covering (minimal energy, maximal polarization...)

Theorem (PSBT for Packing)

Let A be a smooth d-manifold embedded in $\mathbb{R}^p$. Let $\mathcal{H}_d$ denote the d-dimensional Hausdorff measure (just surface measure for A in this case.) Then

$$\lim_{N\to\infty} \delta(A;N)N^{-1/d} = C_d\mathcal{H}_d(A)^{1/d},$$

where C_d is a positive constant depending on only d.

PSBT for Packing

. . . Moreover, if $\mathcal{H}_d(A) > 0$ *then the sequence of probability measures*

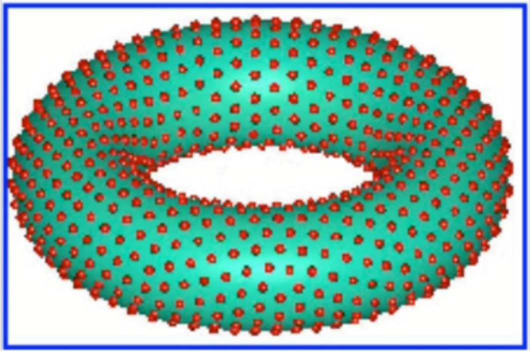
$$\nu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$$

obtained from best packing configurations $\omega_N := \{x_1, \ldots, x_N\}, N \geq 2$ *is uniformly distributed with respect to normalized* $\mathcal{H}_d$ *in the limit. That is, in the weak* topology of measures*

$$\nu_N \xrightarrow{*} \frac{\mathcal{H}_d(A \cap \cdot)}{\mathcal{H}_d(A)}, N \to \infty.$$

The Torus (Bagel)

- $Cd = (\Delta_d/\beta_d)^{1/d}$ where β_d is the volume of the unit ball in $\mathbb{R}^d$, Δ_d is the the largest sphere packing density in $\mathbb{R}^d$
- PSBT Part 2 $\implies$ for the torus, $A := T \subset \mathbb{R}^3$ $(d = 2)$ best packing configurations uniformly distributed for large N



Covering Asymptotics

- 'PSBT' for covering (circa 1950-1960 by Kolmogorov & Tichomirov) only proven for **Jordan measurable** sets in the unconstrained case, for full dimension, with nothing about distribution.

Theorem (PSBT for Unconstrained Covering)

Let $A \subset \mathbb{R}^p$ be a compact set with $\mathcal{L}_p(\partial A) = 0$. Define $\Sigma_p^* := \inf_{N \geq 1} \eta^*([0,1]^p; N) \cdot N^{1/p}$. Then $\Sigma_p^* > 0$ and

$$\lim_{n \to \infty} \eta^*(A; N) \cdot N^{1/p} = \Sigma_p^* \mathcal{L}_p(A)^{1/p}.$$

Corollary 1

- 'Existence of unconstrained covering asymptotics $\implies$ existence of constrained covering asymptotics, and the limits agree' so we get the following...

Theorem (PSBT for Constrained Covering)

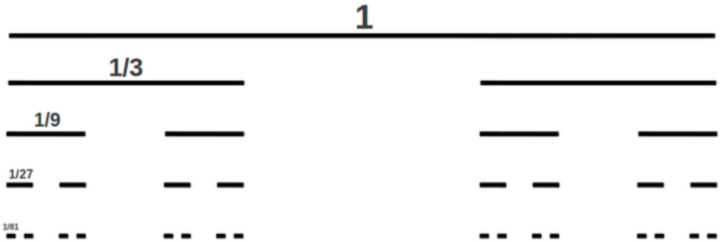
Let $A \subset \mathbb{R}^p$ be a compact set with $\mathcal{L}_p(\partial A) = 0$. Then

$$\lim_{n \to \infty} \eta(A; N) \cdot N^{1/p} = \Sigma_p^* \mathcal{L}_p(A)^{1/p}.$$

- PSBT for Packing is known under **very mild smoothness assumptions** (e.g. on sets that are the image of a compact set under a Lipshitz map). This should happen for covering as well...

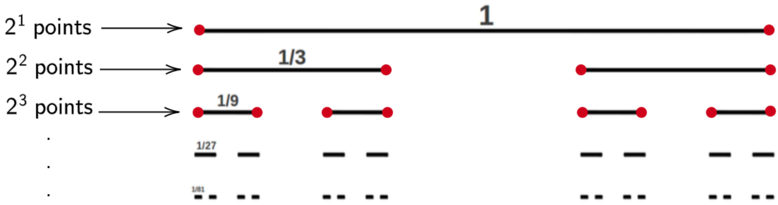
How Much Regularity is Necessary? Enter Fractals

- Without some "rectifiability" what can be said? Fractals generally lack any smoothness but possess nice scaling and translation properties.
- We begin by examining packing on the 1/3-**Cantor Set**.



Asymptotics of Best Packing on the 1/3-Cantor Set

2^n point best packings behave nicely on the 1/3-Cantor set C.



$$\delta(C;2^n) = 1/3^{n-1},\ n \geqslant 1$$

$$\delta(C;2^n) = 1/3^{n-1}, \text{ and } \dim_H(C) = d = \log_3(2)$$
$$\Longrightarrow \lim_{n\to\infty} \delta(C;2^n)\cdot 2^{n/d} = \lim_{n\to\infty} \frac{3^n}{3^{n-1}} = 3$$

- So the limit at least exists along a *subsequence*!

Breakdown of PSBT for Packing on the Cantor Set

On the $1/3$-Cantor set C we obtain

$$\delta(C; 2^n + 1) = 1/3^n \implies \limsup_{N\to\infty} \delta(C; N) \cdot N^{1/d} > \liminf_{N\to\infty} \delta(C; N) \cdot N^{1/d}$$

- So $\lim_{N\to\infty} \delta(C; N) \cdot N^{1/d}$ **does not exist**.
- Does this phenomenon hold on more general fractals?
- What is the story for covering?

Self-Similiar Fractals

- A compact set $A \subset \mathbb{R}^p$ is a **self-similar fractal** fixed under K similitudes if

$$A = \bigcup_{j=1}^{K} \psi_j(A)$$

  where the **similitudes** ψ_j are contraction mappings satisfying $\psi_j(x) = r_j \mathcal{O}(x) + z_j$, for **contraction ratios** $0 < r_j < 1$, and an orthogonal matrix $\mathcal{O} \in \mathcal{O}(p)$
- Our self-similar fractals must also satisfy the **Open Set Condition (OSC)** and have $\psi_j(A) \cap \psi_i(A) = \emptyset, i \neq j$.

Dependent vs. Independent Similitudes

- For a self similar fractal A with similitudes $\{\psi_j\}_{j=1}^K$ and contraction ratios $\{r_j\}_{j=1}^K$, the similitudes are **dependent** (or lattice) if there exists $h > 0$ such that

$$\left\{ \sum_{j=1}^{M} a_j \log(r_j) : a_j \in \mathbb{Z} \right\} = h\mathbb{Z}.$$

- Otherwise the similitudes are **independent**
- The 1/3-Cantor set $C \subset \mathbb{R}^1$ is a dependent self-similar fractal with similitudes $\psi_1(x) = 1/3x$, and $\psi_2(x) = 1/3x + 2/3$.

Packing and Covering on Self-Similar Fractals

- Let $N(\epsilon) := \max\{n \geq 2 : \delta(A; n) \geq \epsilon\}$, $M(\epsilon) := \min\{n \geq 2 : \eta(A; n) \leq \epsilon\}$, $\epsilon > 0$

Theorem (PSBT for Packing and Constrained Covering on Self-similar fractals)

Suppose A is a self-similar fractal with $\dim_H(A) = d$. Then the following limits

$$\lim_{N\to\infty} \delta(A; N) \cdot N^{1/d} = \lim_{\epsilon\to 0+} \epsilon \cdot N(\epsilon)^{1/d}$$

$$\lim_{M\to\infty} \eta(A; M) \cdot M^{1/d} = \lim_{\epsilon\to 0+} \epsilon \cdot M(\epsilon)^{1/d}$$

exist if and only if the similitudes defining A are independent.

Proof Sketch

1. Independent $\implies$ existence: due to S.P. Lalley and **Renewal Theory** from Probability
2. Dependent $\implies$ non-existence: For some $0 < r < 1$, contractions for A look like $\{r^{i_j}\}_{j=1}^{K}, i_j \in \mathbb{Z}^+$ relatively prime
   - $\delta(A; N(r^n)) \geq r^n$ and $\delta(A; N(r^n) + 1) \leq Cr^n$ for some $C < 1$, n large
   - $\eta(A; M(r^n)) \leq r^n$ and $\eta(A; M(r^n) - 1) \geq Dr^n$ for some $D > 1$, n large

Corollary 2

- Let $M^*(\epsilon) := \min\{n \geq 2 : \eta^*(A; n) \leq \epsilon\}$, $\epsilon > 0$
- By contrapositive, '**Non-existence** of constrained covering asymptotics $\implies$ **non-existence** of unconstrained covering asymptotics' so we get the following...

Theorem (Unconstrained Covering on Dependent Fractals)

Suppose A is a dependent self-similar fractal with $\dim_H(A) = d$. Then the following limits do not exist

$$\lim_{M \to \infty} \eta^*(A; M) \cdot M^{1/d}$$

$$\lim_{\epsilon \to 0+} \epsilon \cdot M^*(\epsilon)^{1/d}$$

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